Problem 17

In each of Problems 13 through 18, find the Laplace transform of the given function.

$$f(t) = (t-3)u_2(t) - (t-2)u_3(t)$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

For the provided function in particular,

$$\begin{aligned} f(t) &= (t-3)u_2(t) - (t-2)u_3(t) \\ &= (t-3)H(t-2) - (t-2)H(t-3), \end{aligned}$$

we have

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}[(t-3)H(t-2) - (t-2)H(t-3)]\, dt \\ &= \int_0^\infty e^{-st}(t-3)H(t-2)\, dt - \int_0^\infty e^{-st}(t-2)H(t-3)\, dt \\ &= \int_2^\infty e^{-st}(t-3)\, dt - \int_3^\infty e^{-st}(t-2)\, dt \\ &= \int_2^\infty te^{-st}\, dt - 3\int_2^\infty e^{-st}\, dt - \int_3^\infty te^{-st}\, dt + 2\int_3^\infty e^{-st}\, dt \\ &= \int_2^\infty \left(-\frac{\partial}{\partial s}e^{-st}\right) dt - 3\int_2^\infty e^{-st}\, dt - \int_3^\infty \left(-\frac{\partial}{\partial s}e^{-st}\right) dt + 2\int_3^\infty e^{-st}\, dt \\ &= -\frac{d}{ds}\int_2^\infty e^{-st}\, dt - 3\int_2^\infty e^{-st}\, dt + \frac{d}{ds}\int_3^\infty e^{-st}\, dt + 2\int_3^\infty e^{-st}\, dt \\ &= -\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\right)\Big|_2^\infty - 3\left(-\frac{1}{s}e^{-st}\right)\Big|_2^\infty + \frac{d}{ds}\left(-\frac{1}{s}e^{-st}\right)\Big|_3^\infty + 2\left(-\frac{1}{s}e^{-st}\right)\Big|_3^\infty \\ &= -\frac{d}{ds}\left(\frac{1}{s}e^{-2s}\right) - 3\left(\frac{1}{s}e^{-2s}\right) + \frac{d}{ds}\left(\frac{1}{s}e^{-3s}\right) + 2\left(\frac{1}{s}e^{-3s}\right) \\ &= -\left(-\frac{1}{s^2}e^{-2s} - \frac{2}{s}e^{-2s}\right) - 3\left(\frac{1}{s}e^{-2s}\right) + \left(-\frac{1}{s^2}e^{-3s} - \frac{3}{s}e^{-3s}\right) + 2\left(\frac{1}{s}e^{-3s}\right) \\ &= \frac{e^{-2s} + 2se^{-2s}}{s^2} - \frac{3se^{-2s}}{s^2} - \frac{e^{-3s} + 3se^{-3s}}{s^2} + \frac{2se^{-3s}}{s^2} \\ &= \frac{e^{-2s} - se^{-2s} - e^{-3s} - se^{-3s}}{s^2} \\ &= \frac{(1-s)e^{-2s} - (1+s)e^{-3s}}{s^2}. \end{aligned}$$