Problem 25

Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.

(a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right), \qquad s > ca.$$

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right).$$

(c) Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$$

Solution

The Laplace transform of a function $f(t)$ is defined to be

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt.$$

Part (a)

Suppose we use $f(ct)$ rather than $f(t)$.

$$\mathcal{L}\{f(ct)\} = \int_0^\infty e^{-st}f(ct)\,dt.$$

Make the substitution $x = ct$. Then $dx = c\,dt$.

$$\begin{aligned}\mathcal{L}\{f(ct)\} &= \int_0^\infty e^{-sx/c}f(x)\left(\frac{dx}{c}\right)\\ &= \frac{1}{c}\int_0^\infty e^{-(s/c)x}f(x)\,dx\\ &= \frac{1}{c}F\left(\frac{s}{c}\right)\end{aligned}$$

Part (b)

Suppose we use $(1/k)f(t/k)$ rather than $f(t)$.

$$\mathcal{L}\left\{\frac{1}{k}f\left(\frac{t}{k}\right)\right\} = \int_0^\infty e^{-st}\frac{1}{k}f\left(\frac{t}{k}\right)dt$$

Make the substitution $x = t/k$. Then $dx = dt/k$.

$$\begin{aligned}\mathcal{L}\left\{\frac{1}{k}f\left(\frac{t}{k}\right)\right\} &= \int_0^\infty e^{-s(kx)}\frac{1}{k}f(x)(k\,dx) \\ &= \int_0^\infty e^{-(ks)x}f(x)\,dx \\ &= F(ks)\end{aligned}$$

Therefore, taking the inverse Laplace transform of both sides,

$$\frac{1}{k}f\left(\frac{t}{k}\right) = \mathcal{L}^{-1}\{F(ks)\}.$$

Part (c)

Suppose we use $(1/a)e^{-bt/a}f(t/a)$ rather than $f(t)$.

$$\mathcal{L}\left\{\frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right)\right\} = \int_0^\infty e^{-st}\frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right)dt$$

Make the substitution $x = t/a$. Then $dx = dt/a$.

$$\begin{aligned}\mathcal{L}\left\{\frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right)\right\} &= \int_0^\infty e^{-s(ax)}\frac{1}{a}e^{-bx}f(x)(a\,dx) \\ &= \int_0^\infty e^{-(as+b)x}f(x)\,dx \\ &= F(as+b)\end{aligned}$$

Therefore, taking the inverse Laplace transform of both sides,

$$\frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right) = \mathcal{L}^{-1}\{F(as+b)\}.$$