Problem 36

In each of Problems 35 through 38, use the result of Problem 34 to find the Laplace transform of the given function.

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2; \end{cases}$$

$f(t+2) = f(t)$.

See Figure 6.3.8.

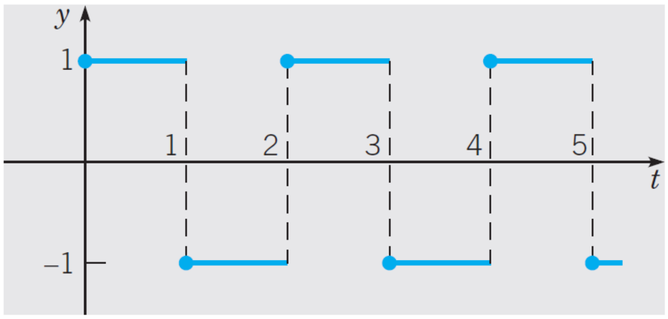

FIGURE 6.3.8 The function $f(t)$ in Problem 36; a square wave.

Solution

For a function that repeats itself periodically every T units, the Laplace transform is

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$

In this problem the period is $T = 2$. Therefore,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{\displaystyle\int_0^2 e^{-st} f(t)\,dt}{1 - e^{-2s}} \\
&= \frac{\displaystyle\int_0^1 e^{-st}(1)\,dt + \int_1^2 e^{-st}(-1)\,dt}{1 - e^{-2s}} \\
&= \frac{\displaystyle\int_0^1 e^{-st}\,dt - \int_1^2 e^{-st}\,dt}{1 - e^{-2s}} \\
&= \frac{\left(-\frac{1}{s}e^{-st}\right)\Big|_0^1 - \left(-\frac{1}{s}e^{-st}\right)\Big|_1^2}{1 - e^{-2s}} \\
&= \frac{\frac{1}{s} - \frac{1}{s}e^{-s} - \left(\frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}\right)}{1 - e^{-2s}} \\
&= \frac{1 - 2e^{-s} + e^{-2s}}{s(1 - e^{-2s})} = \frac{(1 - e^{-s})^2}{s(1 + e^{-s})(1 - e^{-s})} = \frac{1 - e^{-s}}{s(1 + e^{-s})}.
\end{aligned}$$