Problem 40

Consider the function p defined by

$$p(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2; \end{cases} \qquad p(t+2) = p(t).$$

(a) Sketch the graph of $y = p(t)$.

(b) Find $\mathcal{L}\{p(t)\}$ by noting that p is the periodic extension of the function h in Problem 39(c) and then using the result of Problem 34.

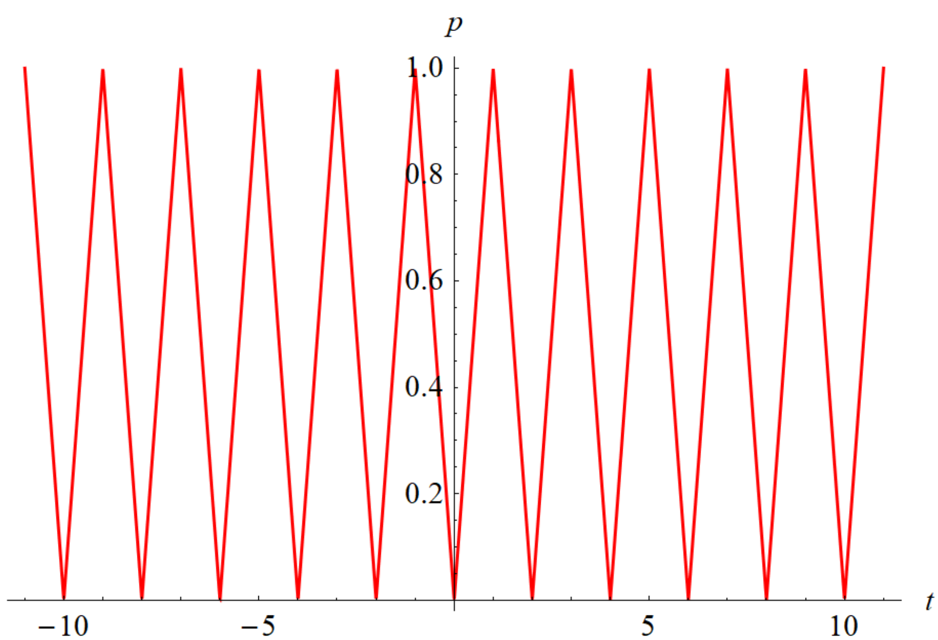
(c) Find $\mathcal{L}\{p(t)\}$ by noting that

$$p(t) = \int_0^t f(t)\,dt,$$

where f is the function in Problem 36, and then using Theorem 6.2.1.

Solution

Part (a)



Part (b)

Since $p(t)$ is periodic, we can use the formula derived in Problem 34 to determine its Laplace transform. The period here is $T = 2$.

$$\begin{aligned}
\mathcal{L}\{p(t)\} &= \frac{\int_0^T e^{-st}p(t)\,dt}{1-e^{-sT}} \\
&= \frac{\int_0^2 e^{-st}p(t)\,dt}{1-e^{-2s}} \\
&= \frac{\int_0^1 e^{-st}(t)\,dt + \int_1^2 e^{-st}(2-t)\,dt}{1-e^{-2s}} \\
&= \frac{\int_0^1 te^{-st}\,dt + 2\int_1^2 e^{-st}\,dt - \int_1^2 te^{-st}\,dt}{1-e^{-2s}} \\
&= \frac{\int_0^1 \left(-\frac{\partial}{\partial s}e^{-st}\right)dt + 2\int_1^2 e^{-st}\,dt - \int_1^2 \left(-\frac{\partial}{\partial s}e^{-st}\right)dt}{1-e^{-2s}} \\
&= \frac{-\frac{d}{ds}\int_0^1 e^{-st}\,dt + 2\int_1^2 e^{-st}\,dt + \frac{d}{ds}\int_1^2 e^{-st}\,dt}{1-e^{-2s}} \\
&= \frac{-\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\Big|_0^1\right) + 2\left(-\frac{1}{s}e^{-st}\Big|_1^2\right) + \frac{d}{ds}\left(-\frac{1}{s}e^{-st}\Big|_1^2\right)}{1-e^{-2s}} \\
&= \frac{-\frac{d}{ds}\left(\frac{1}{s}-\frac{1}{s}e^{-s}\right) + 2\left(\frac{1}{s}e^{-s}-\frac{1}{s}e^{-2s}\right) + \frac{d}{ds}\left(\frac{1}{s}e^{-s}-\frac{1}{s}e^{-2s}\right)}{1-e^{-2s}} \\
&= \frac{-\left(-\frac{1}{s^2}+\frac{1}{s^2}e^{-s}+\frac{1}{s}e^{-s}\right) + 2\left(\frac{1}{s}e^{-s}-\frac{1}{s}e^{-2s}\right) + \left(-\frac{1}{s^2}e^{-s}-\frac{1}{s}e^{-s}+\frac{1}{s^2}e^{-2s}+\frac{2}{s}e^{-2s}\right)}{1-e^{-2s}} \\
&= \frac{\frac{1}{s^2}+\frac{e^{-2s}}{s^2}-\frac{2e^{-s}}{s^2}}{1-e^{-2s}} \\
&= \frac{1-2e^{-s}+e^{-2s}}{s^2(1-e^{-2s})} \\
&= \frac{(1-e^{-s})^2}{s^2(1+e^{-s})(1-e^{-s})} \\
&= \frac{1-e^{-s}}{s^2(1+e^{-s})}
\end{aligned}$$

Part (c)

$$p(t) = \int_0^t f(t)\,dt$$

Differentiate both sides with respect to t.

$$\frac{dp}{dt} = f(t)$$

Take the Laplace transform of both sides.

$$\mathcal{L}\left\{\frac{dp}{dt}\right\} = \mathcal{L}\{f(t)\}$$

Use the fact that the derivative transforms as $\mathcal{L}\{p'(t)\} = s\mathcal{L}\{p(t)\} - p(0)$. This is Theorem 6.2.1 in the textbook.

$$s\mathcal{L}\{p(t)\} - p(0) = \mathcal{L}\{f(t)\}$$

Solve for $\mathcal{L}\{p(t)\}$.

$$\mathcal{L}\{p(t)\} = \frac{\mathcal{L}\{f(t)\} + p(0)}{s}$$

It was found in Problem 36 that

$$\mathcal{L}\{f(t)\} = \frac{1 - e^{-s}}{s(1 + e^{-s})}.$$

Therefore, since $p(0) = 0$,

$$\begin{aligned}\mathcal{L}\{p(t)\} &= \frac{\frac{1-e^{-s}}{s(1+e^{-s})} + 0}{s} \\ &= \frac{1 - e^{-s}}{s^2(1 + e^{-s})}.\end{aligned}$$