Problem 5

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + 3y' + 2y = f(t); \qquad y(0) = 0, \quad y'(0) = 0; \qquad f(t) = \begin{cases} 1, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{f(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2[Y(s)] = \int_0^{10} e^{-st}(1)\,dt + \int_{10}^\infty e^{-st}(0)\,dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 3[sY(s)] + 2[Y(s)] = \int_0^{10} e^{-st}\,dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 3s + 2)Y(s) = \left(-\frac{1}{s}e^{-st}\right)\Big|_0^{10}$$

$$(s^2 + 3s + 2)Y(s) = \frac{1}{s} - \frac{1}{s}e^{-10s}$$

Solve for $Y(s)$ and write it in terms of known transforms by using partial fraction decomposition.

$$\begin{aligned} Y(s) &= \frac{1}{s(s^2+3s+2)} - \frac{1}{s(s^2+3s+2)}e^{-10s} \\ &= \frac{1}{s(s+1)(s+2)} - \frac{1}{s(s+1)(s+2)}e^{-10s} \\ &= \frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2} - \left(\frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)e^{-10s} \end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2} - \left(\frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)e^{-10s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}\right\} - \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)e^{-10s}\right\} \\ &= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - \left[\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right]H(t-10) \\ &= \frac{1}{2}(1 - 2e^{-t} + e^{-2t}) - \frac{1}{2}[(1 - 2e^{-(t-10)} + e^{-2(t-10)}]H(t-10) \\ &= \frac{1}{2}(1-e^{-t})^2 - \frac{1}{2}[1-e^{-(t-10)}]^2 H(t-10) \\ &= \frac{1}{2}(1-e^{-t})^2 - \frac{1}{2}[1-e^{-(t-10)}]^2 u_{10}(t) \end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $f(t)$ versus t.