Problem 6

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + 3y' + 2y = u_2(t); \qquad y(0) = 0, \quad y'(0) = 1$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{u_2(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{u_2(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2[Y(s)] = \int_0^\infty e^{-st}[u_2(t)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$[s^2Y(s) - 1] + 3[sY(s)] + 2[Y(s)] = \int_2^\infty e^{-st}\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 3s + 2)Y(s) - 1 = \left(-\frac{1}{s}e^{-st}\right)\Big|_2^\infty$$
$$= \frac{1}{s}e^{-2s}$$

$$(s^2 + 3s + 2)Y(s) = \frac{1}{s}e^{-2s} + 1$$

Solve for $Y(s)$ and write it in terms of known transforms by using partial fraction decomposition.

$$\begin{aligned}
Y(s) &= \frac{1}{s(s^2+3s+2)}e^{-2s} + \frac{1}{s^2+3s+2} \\
&= \frac{1}{s(s+1)(s+2)}e^{-2s} + \frac{1}{(s+1)(s+2)} \\
&= \left(\frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)e^{-2s} + \frac{1}{s+1} + \frac{-1}{s+2}
\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)e^{-2s} + \frac{1}{s+1} + \frac{-1}{s+2}\right\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)e^{-2s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\
&= \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]H(t-2) + e^{-t} - e^{-2t} \\
&= \frac{1}{2}[1 - 2e^{-(t-2)} + e^{-2(t-2)}]H(t-2) + e^{-t} - e^{-2t} \\
&= \frac{1}{2}[1 - e^{-(t-2)}]^2 H(t-2) + e^{-t} - e^{-2t} \\
&= \frac{1}{2}(1 - e^{2-t})^2 u_2(t) + e^{-t} - e^{-2t}
\end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $f(t)$ versus t.