Problem 9

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + y = g(t); \qquad y(0) = 0, \quad y'(0) = 1; \qquad g(t) = \begin{cases} t/2, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{g(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \int_0^6 e^{-st}(t/2)\, dt + \int_6^\infty e^{-st}(3)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$[s^2Y(s) - 1] + Y(s) = \frac{1}{2}\int_0^6 te^{-st}\, dt + 3\int_6^\infty e^{-st}\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$\begin{aligned}(s^2+1)Y(s) - 1 &= \frac{1}{2}\int_0^6 \left(-\frac{\partial}{\partial s}e^{-st}\right) dt + 3\int_6^\infty e^{-st}\, dt \\ &= -\frac{1}{2}\frac{d}{ds}\int_0^6 e^{-st}\, dt + 3\int_6^\infty e^{-st}\, dt \\ &= -\frac{1}{2}\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\bigg|_0^6\right) + 3\left(-\frac{1}{s}e^{-st}\bigg|_6^\infty\right)\end{aligned}$$

Finish simplifying the right side.

$$\begin{aligned}(s^2+1)Y(s)-1 &= -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}-\frac{1}{s}e^{-6s}\right)+3\left(\frac{1}{s}e^{-6s}\right)\\ &= -\frac{1}{2}\left(-\frac{1}{s^2}+\frac{1}{s^2}e^{-6s}+\frac{6}{s}e^{-6s}\right)+\frac{3}{s}e^{-6s}\\ &= \frac{1/2}{s^2}-\frac{1/2}{s^2}e^{-6s}\end{aligned}$$

Solve for $Y(s)$.

$$\begin{aligned}Y(s) &= \frac{1/2}{s^2(s^2+1)}-\frac{1/2}{s^2(s^2+1)}e^{-6s}+\frac{1}{s^2+1}\\ &= \left(\frac{\frac{1}{2}}{s^2}-\frac{\frac{1}{2}}{s^2+1}\right)-\left(\frac{\frac{1}{2}}{s^2}-\frac{\frac{1}{2}}{s^2+1}\right)e^{-6s}+\frac{1}{s^2+1}\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{Y(s)\}\\ &= \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{2}}{s^2}-\frac{\frac{1}{2}}{s^2+1}\right)-\left(\frac{\frac{1}{2}}{s^2}-\frac{\frac{1}{2}}{s^2+1}\right)e^{-6s}+\frac{1}{s^2+1}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s^2}-\frac{\frac{1}{2}}{s^2+1}\right\}-\mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{2}}{s^2}-\frac{\frac{1}{2}}{s^2+1}\right)e^{-6s}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= \frac{1}{2}t-\frac{1}{2}\sin t-\left[\frac{1}{2}(t-6)-\frac{1}{2}\sin(t-6)\right]H(t-6)+\sin t\\ &= \frac{1}{2}(t+\sin t)-\frac{1}{2}\left[(t-6)-\sin(t-6)\right]H(t-6)\\ &= \frac{1}{2}(t+\sin t)-\frac{1}{2}\left[(t-6)-\sin(t-6)\right]u_6(t)\end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $g(t)$ versus t.