Problem 11

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + 4y = u_\pi(t) - u_{3\pi}(t); \qquad y(0) = 0, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{u_\pi(t) - u_{3\pi}(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{u_\pi(t)\} - \mathcal{L}\{u_{3\pi}(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 4[Y(s)] = \int_0^\infty e^{-st}[u_\pi(t)]\, dt - \int_0^\infty e^{-st}[u_{3\pi}(t)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 4[Y(s)] = \int_\pi^\infty e^{-st}\, dt - \int_{3\pi}^\infty e^{-st}\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 4)Y(s) = \left(-\frac{1}{s}e^{-st}\right)\Big|_\pi^\infty - \left(-\frac{1}{s}e^{-st}\right)\Big|_{3\pi}^\infty$$
$$= \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-3\pi s}$$

Solve for $Y(s)$ and write the right side in terms of known transforms.

$$Y(s) = \frac{1}{s(s^2 + 4)}e^{-\pi s} - \frac{1}{s(s^2 + 4)}e^{-3\pi s}$$

Use partial fraction decomposition.

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

Multiply both sides by $s(s^2+4)$.

$$1 = A(s^2+4) + (Bs+C)s$$

Plug in three random values of s to get a system of three equations for A, B, and C.

$$\begin{aligned} s = 0: &\quad 1 = 4A \\ s = 1: &\quad 1 = 5A + B + C \\ s = 2: &\quad 1 = 8A + 4B + 2C \end{aligned}$$

Solving this system yields $A = 1/4$, $B = -1/4$, and $C = 0$.

$$Y(s) = \left(\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s}{s^2+4}\right)e^{-\pi s} - \left(\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s}{s^2+4}\right)e^{-3\pi s}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s}{s^2+4}\right)e^{-\pi s} - \left(\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s}{s^2+4}\right)e^{-3\pi s}\right\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s}{s^2+4}\right)e^{-\pi s}\right\} - \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s}{s^2+4}\right)e^{-3\pi s}\right\} \\ &= \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-\pi)\right]H(t-\pi) - \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-3\pi)\right]H(t-3\pi) \\ &= \frac{1}{4}(1-\cos 2t)H(t-\pi) - \frac{1}{4}(1-\cos 2t)H(t-3\pi) \\ &= \frac{1}{4}(1-\cos 2t)[H(t-\pi) - H(t-3\pi)] \\ &= \frac{1}{4}[1-(1-2\sin^2 t)][H(t-\pi) - H(t-3\pi)] \\ &= \frac{1}{2}\sin^2 t[H(t-\pi) - H(t-3\pi)] \\ &= \frac{1}{2}\sin^2 t[u_\pi(t) - u_{3\pi}(t)] \end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $f(t)$ versus t.