Problem 12

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y^{(4)} - y = u_1(t) - u_2(t); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the derivatives transform as follows.

$$\begin{aligned}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)
\end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y^{(4)} - y\} = \mathcal{L}\{u_1(t) - u_2(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = \mathcal{L}\{u_1(t)\} - \mathcal{L}\{u_2(t)\}$$

$$[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] - [Y(s)] = \int_0^\infty e^{-st}[u_1(t)]\, dt - \int_0^\infty e^{-st}[u_2(t)]\, dt$$

Plug in the initial conditions, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$, and $y'''(0) = 0$.

$$[s^4Y(s)] - [Y(s)] = \int_1^\infty e^{-st}\, dt - \int_2^\infty e^{-st}\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$\begin{aligned}
(s^4 - 1)Y(s) &= \left(-\frac{1}{s}e^{-st}\right)\Bigg|_1^\infty - \left(-\frac{1}{s}e^{-st}\right)\Bigg|_2^\infty \\
&= \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}
\end{aligned}$$

Solve for $Y(s)$ and write the right side in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{1}{s(s^4-1)}e^{-s} - \frac{1}{s(s^4-1)}e^{-2s} \\ &= \frac{1}{s(s-1)(s+1)(s^2+1)}e^{-s} - \frac{1}{s(s-1)(s+1)(s^2+1)}e^{-2s} \end{aligned}$$

Use partial fraction decomposition.

$$\frac{1}{s(s-1)(s+1)(s^2+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{Ds+E}{s^2+1}$$

Multiply both sides by $s(s-1)(s+1)(s^2+1)$.

$$1 = A(s-1)(s+1)(s^2+1) + Bs(s+1)(s^2+1) + Cs(s-1)(s^2+1) + (Ds+E)s(s-1)(s+1)$$

Plug in five random values of s to get a system of five equations for A, B, C, D, and E.

$$\begin{aligned} s=0: &\quad 1=-A \\ s=1: &\quad 1=4B \\ s=-1: &\quad 1=4C \\ s=2: &\quad 1=15A+30B+10C+12D+6E \\ s=3: &\quad 1=80A+120B+60C+72D+24E \end{aligned}$$

Solving this system yields $A=-1$, $B=1/4$, $C=1/4$, $D=1/2$, and $E=0$.

$$Y(s) = \left(\frac{-1}{s} + \frac{1/4}{s-1} + \frac{1/4}{s+1} + \frac{\frac{1}{2}s}{s^2+1}\right)e^{-s} - \left(\frac{-1}{s} + \frac{1/4}{s-1} + \frac{1/4}{s+1} + \frac{\frac{1}{2}s}{s^2+1}\right)e^{-2s}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{-1}{s} + \frac{1/4}{s-1} + \frac{1/4}{s+1} + \frac{\frac{1}{2}s}{s^2+1}\right)e^{-s} - \left(\frac{-1}{s} + \frac{1/4}{s-1} + \frac{1/4}{s+1} + \frac{\frac{1}{2}s}{s^2+1}\right)e^{-2s}\right\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{-1}{s} + \frac{1/4}{s-1} + \frac{1/4}{s+1} + \frac{\frac{1}{2}s}{s^2+1}\right)e^{-s}\right\} - \mathcal{L}^{-1}\left\{\left(\frac{-1}{s} + \frac{1/4}{s-1} + \frac{1/4}{s+1} + \frac{\frac{1}{2}s}{s^2+1}\right)e^{-2s}\right\} \\ &= \left[-1 + \frac{1}{4}e^{(t-1)} + \frac{1}{4}e^{-(t-1)} + \frac{1}{2}\cos(t-1)\right]H(t-1) - \left[-1 + \frac{1}{4}e^{(t-2)} + \frac{1}{4}e^{-(t-2)} + \frac{1}{2}\cos(t-2)\right]H(t-2) \\ &= \left[-1 + \frac{1}{2}\frac{e^{(t-1)}+e^{-(t-1)}}{2} + \frac{1}{2}\cos(t-1)\right]H(t-1) - \left[-1 + \frac{1}{2}\frac{e^{(t-2)}+e^{-(t-2)}}{2} + \frac{1}{2}\cos(t-2)\right]H(t-2) \\ &= \left[-1 + \frac{1}{2}\cosh(t-1) + \frac{1}{2}\cos(t-1)\right]H(t-1) - \left[-1 + \frac{1}{2}\cosh(t-2) + \frac{1}{2}\cos(t-2)\right]H(t-2) \\ &= \frac{1}{2}\left[-2 + \cosh(t-1) + \cos(t-1)\right]u_1(t) - \frac{1}{2}\left[-2 + \cosh(t-2) + \cos(t-2)\right]u_2(t) \end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $f(t)$ versus t.