Problem 17

Modify the problem in Example 2 of this section by replacing the given forcing function $g(t)$ by

$$f(t) = \left[u_5(t)(t-5) - u_{5+k}(t)(t-5-k)\right]/k.$$

(a) Sketch the graph of $f(t)$ and describe how it depends on k. For what value of k is $f(t)$ identical to $g(t)$ in the example?

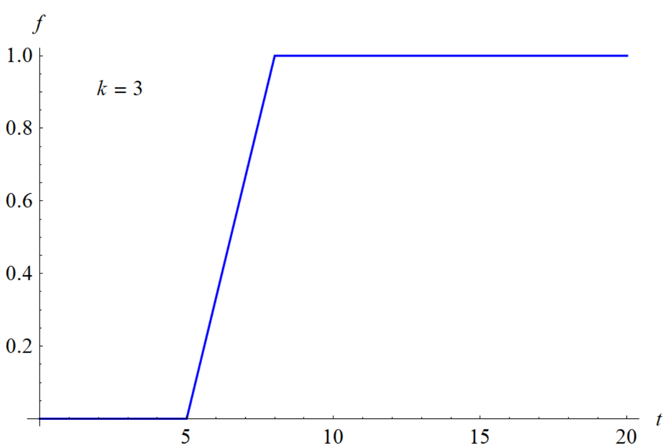
(b) Solve the initial value problem

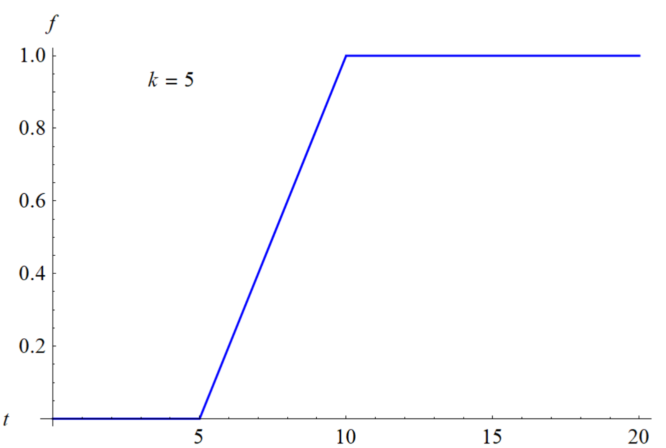
$$y'' + 4y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

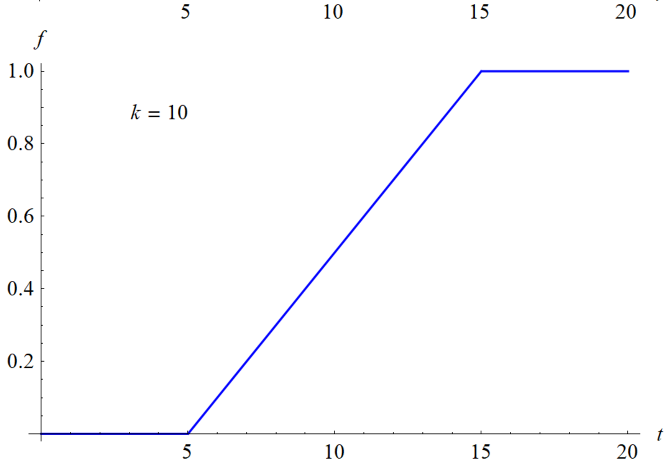
(c) The solution in part (b) depends on k, but for sufficiently large t the solution is always a simple harmonic oscillation about $y = 1/4$. Try to decide how the amplitude of this eventual oscillation depends on k. Then confirm your conclusion by plotting the solution for a few different values of k.

Solution

Part (a)







The value of k determines the horizontal length of the ramp. $f(t)$ is identical to $g(t)$ if $k = 5$.

Part (b)

$$y'' + 4y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned}\mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0)\end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{k}[u_5(t)(t-5) - u_{5+k}(t)(t-5-k)]\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \frac{1}{k}\mathcal{L}\{u_5(t)(t-5)\} - \frac{1}{k}\mathcal{L}\{u_{5+k}(t)(t-5-k)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 4[Y(s)] = \frac{1}{k}\int_0^\infty e^{-st}[u_5(t)(t-5)]\, dt - \frac{1}{k}\int_0^\infty e^{-st}[u_{5+k}(t)(t-5-k)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 4[Y(s)] = \frac{1}{k}\int_5^\infty (t-5)e^{-st}\, dt - \frac{1}{k}\int_{5+k}^\infty (t-5-k)e^{-st}\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2+4)Y(s) = \frac{1}{k}\left(\frac{e^{-5s}}{s^2}\right) - \frac{1}{k}\left[\frac{e^{-(5+k)s}}{s^2}\right]$$

Solve for $Y(s)$ and write the right side in terms of known transforms.

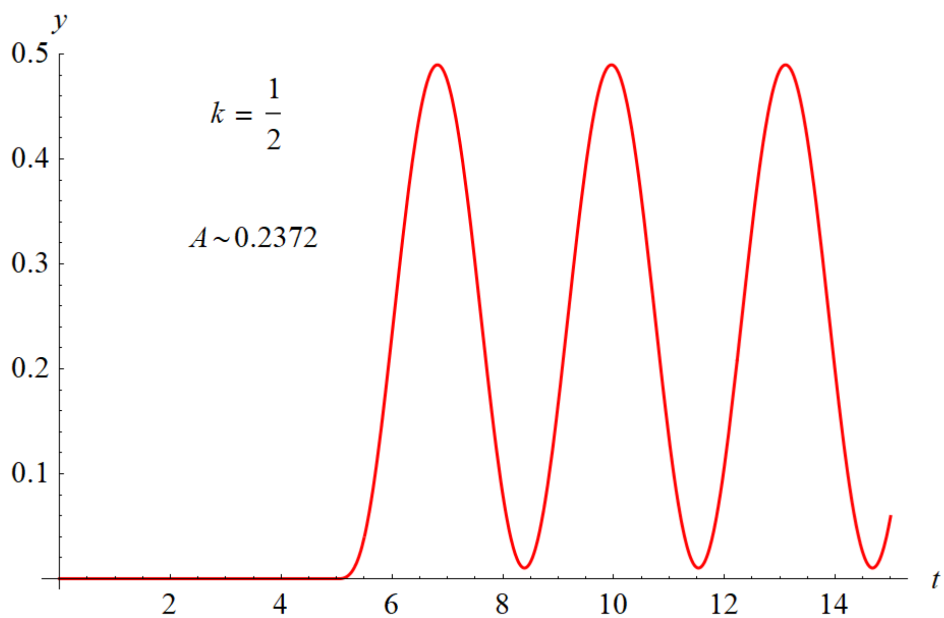
$$\begin{aligned}Y(s) &= \frac{1}{k}\left[\frac{1}{s^2(s^2+4)}e^{-5s} - \frac{1}{s^2(s^2+4)}e^{-(5+k)s}\right] \\ &= \frac{1}{k}\left[\left(\frac{\frac{1}{4}}{s^2} + \frac{-\frac{1}{4}}{s^2+4}\right)e^{-5s} - \left(\frac{\frac{1}{4}}{s^2} + \frac{-\frac{1}{4}}{s^2+4}\right)e^{-(5+k)s}\right] \\ &= \frac{1}{k}\left(\frac{\frac{1}{4}}{s^2} - \frac{1}{8}\frac{2}{s^2+4}\right)e^{-5s} - \frac{1}{k}\left(\frac{\frac{1}{4}}{s^2} - \frac{1}{8}\frac{2}{s^2+4}\right)e^{-(5+k)s}\end{aligned}$$

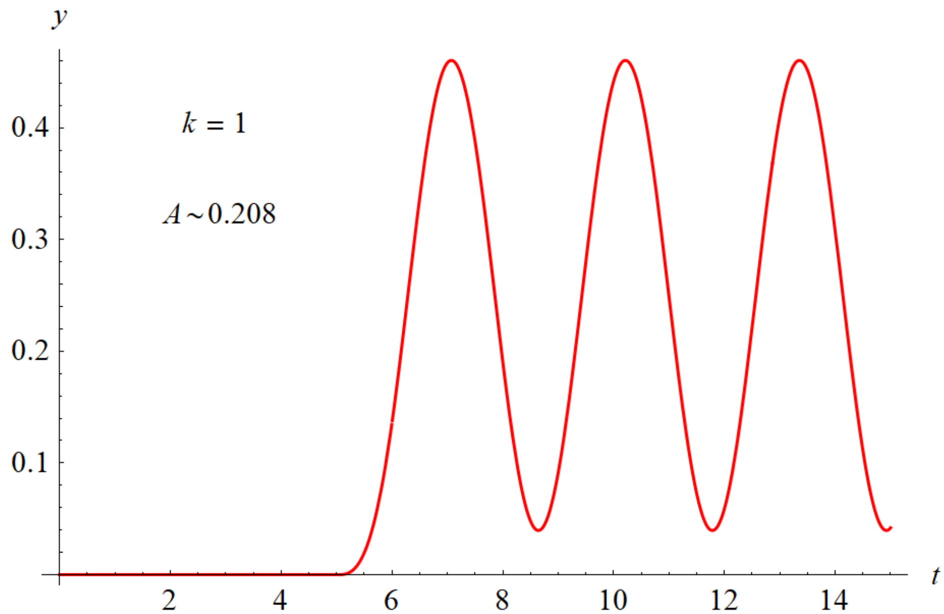
Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

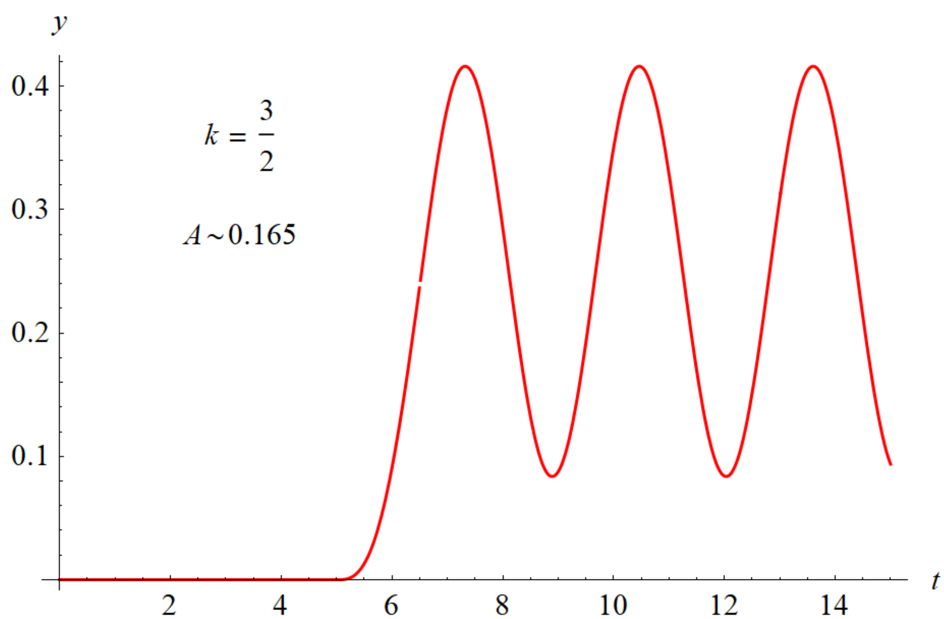
$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{k}\left(\frac{\frac{1}{4}}{s^2} - \frac{1}{8}\frac{2}{s^2+4}\right)e^{-5s} - \frac{1}{k}\left(\frac{\frac{1}{4}}{s^2} - \frac{1}{8}\frac{2}{s^2+4}\right)e^{-(5+k)s}\right\} \\
&= \frac{1}{k}\mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{4}}{s^2} - \frac{1}{8}\frac{2}{s^2+4}\right)e^{-5s}\right\} - \frac{1}{k}\mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{4}}{s^2} - \frac{1}{8}\frac{2}{s^2+4}\right)e^{-(5+k)s}\right\} \\
&= \frac{1}{k}\left\{\frac{1}{4}(t-5) - \frac{1}{8}\sin[2(t-5)]\right\}H(t-5) - \frac{1}{k}\left\{\frac{1}{4}(t-5-k) - \frac{1}{8}\sin[2(t-5-k)]\right\}H(t-5-k) \\
&= \frac{1}{k}\left\{\frac{1}{4}(t-5) - \frac{1}{8}\sin[2(t-5)]\right\}u_5(t) - \frac{1}{k}\left\{\frac{1}{4}(t-5-k) - \frac{1}{8}\sin[2(t-5-k)]\right\}u_{5+k}(t)
\end{aligned}$$

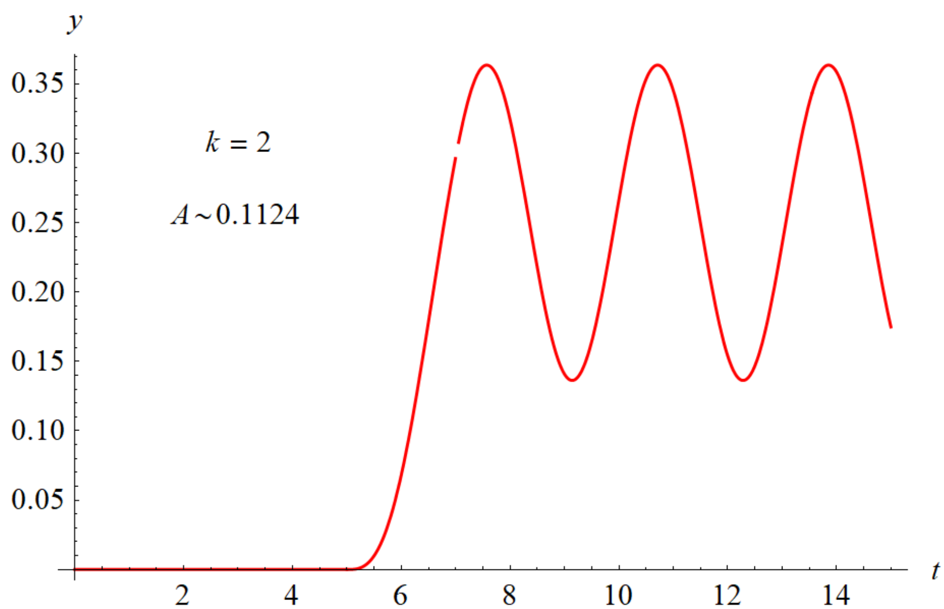
Part (c)

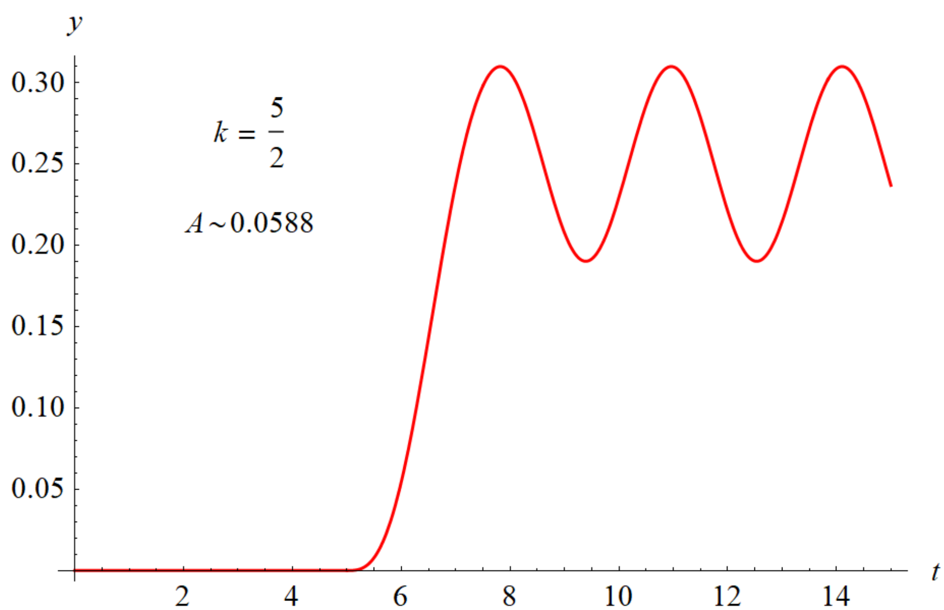
Below are several plots of $y(t)$ versus t for several values of k.

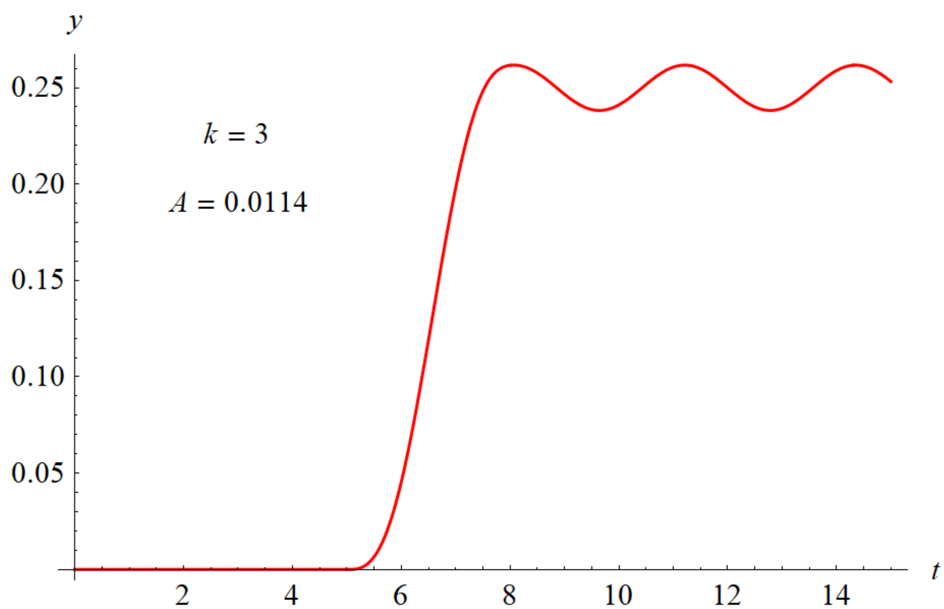


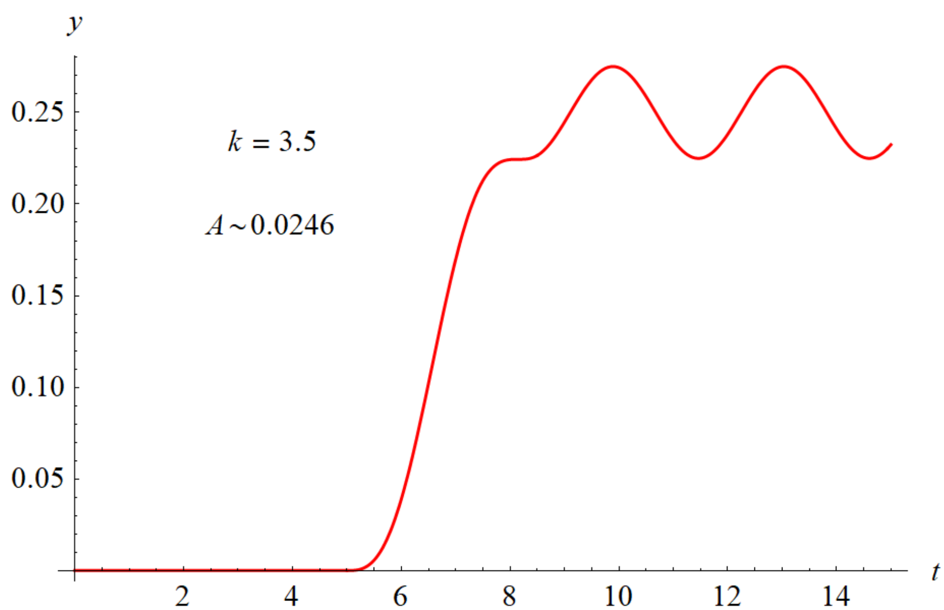

y
0.4
0.3
0.2
0.1
k = 1
A ~ 0.208
2
4
6
8
10
12
14
t

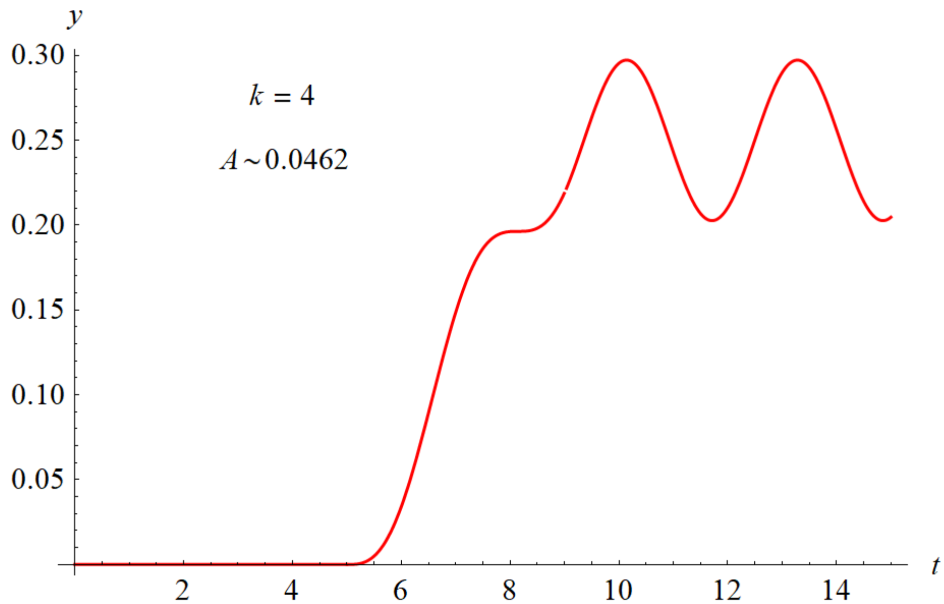

y
0.4
0.3
0.2
0.1
k = 3/2
A ~ 0.165
2
4
6
8
10
12
14
t

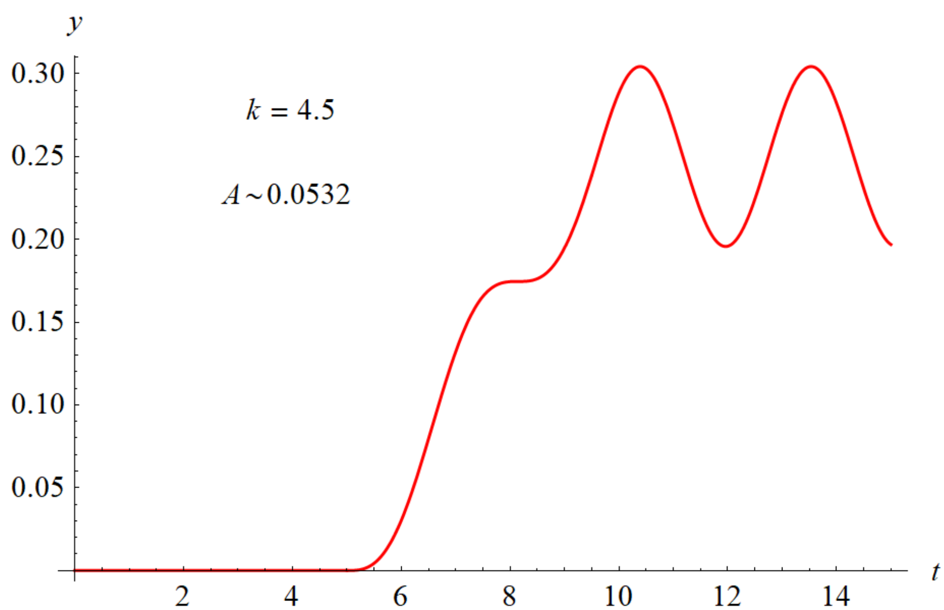

y
0.35
0.30
0.25
0.20
0.15
0.10
0.05
k = 2
A ~ 0.1124
2
4
6
8
10
12
14
t

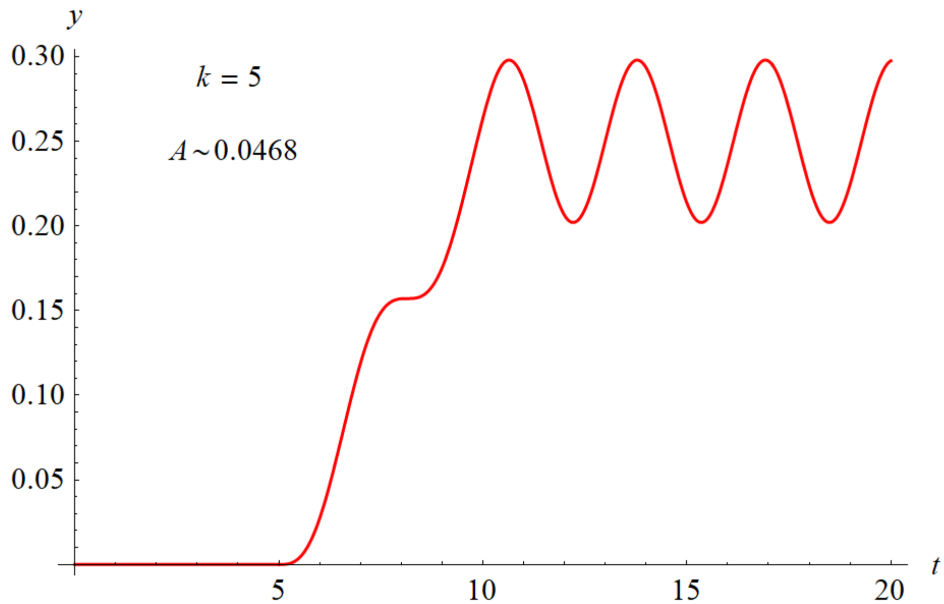

y
0.30
0.25
0.20
0.15
0.10
0.05
k = 5/2
A ~ 0.0588
2
4
6
8
10
12
14
t

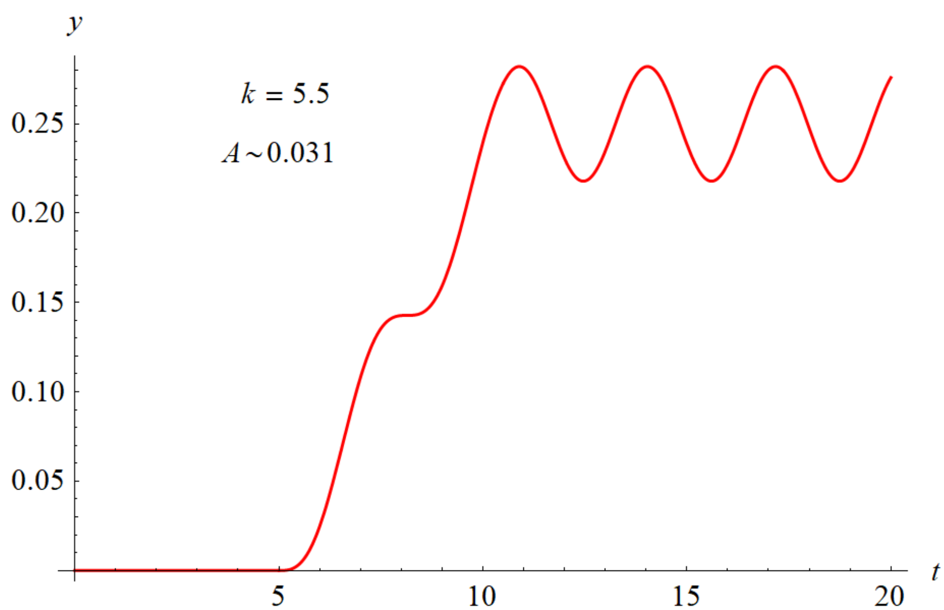

y
0.25
0.20
0.15
0.10
0.05
k = 3
A = 0.0114
2
4
6
8
10
12
14
t

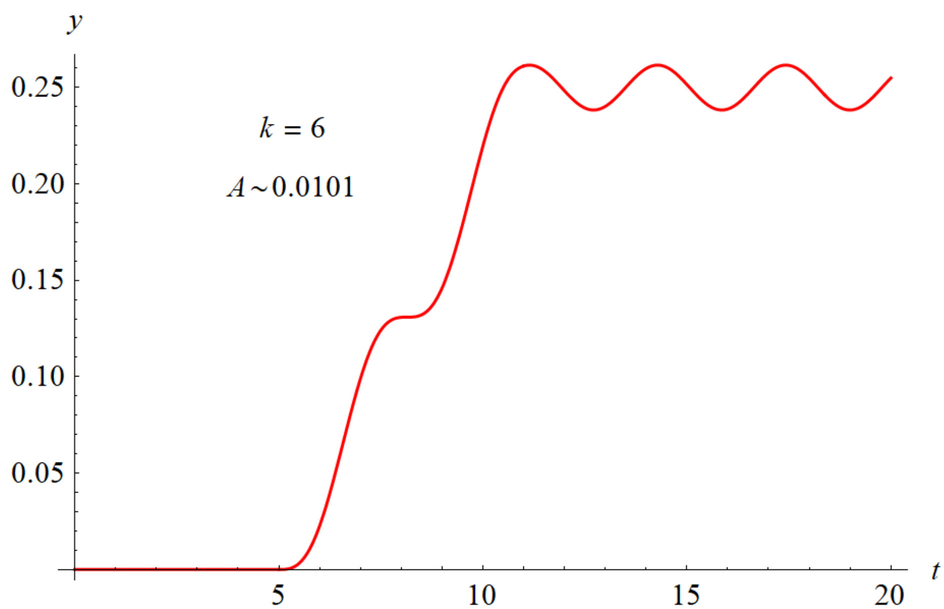

y
0.25
0.20
0.15
0.10
0.05
k = 3.5
A ~ 0.0246
2
4
6
8
10
12
14
t

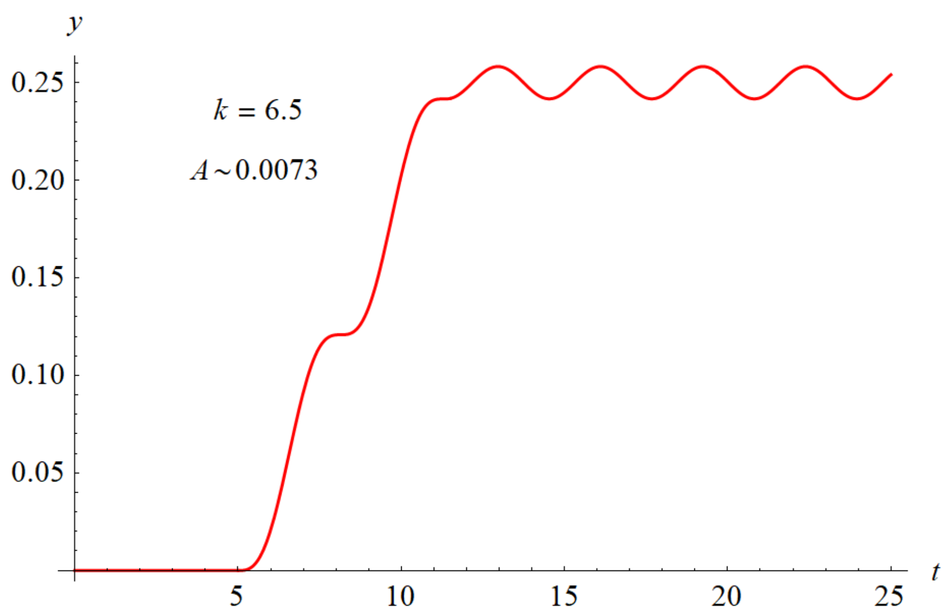

y
0.30
0.25
0.20
0.15
0.10
0.05
k = 4
A ~ 0.0462
2
4
6
8
10
12
14
t

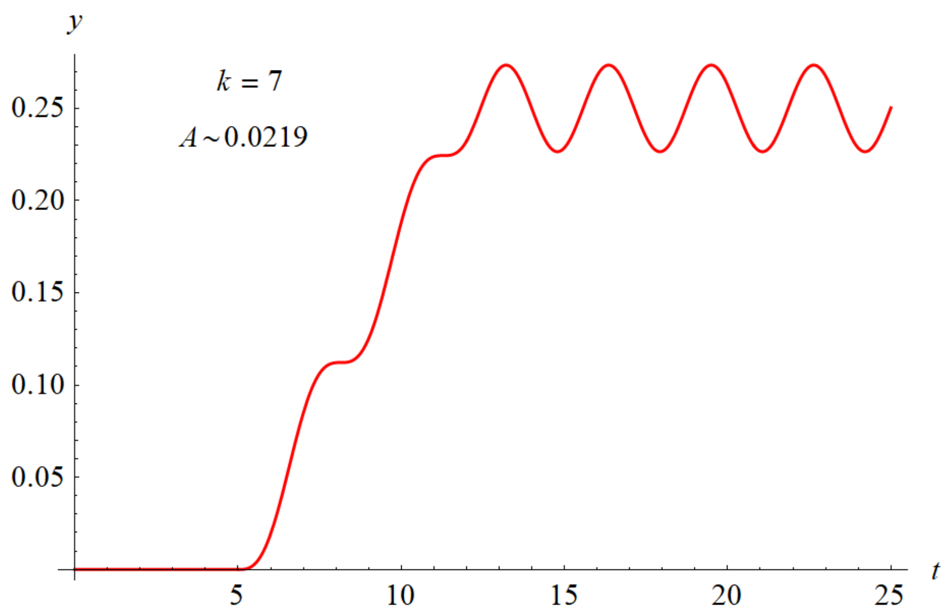

y
0.30
0.25
0.20
0.15
0.10
0.05
k = 4.5
A ~ 0.0532
2
4
6
8
10
12
14
t

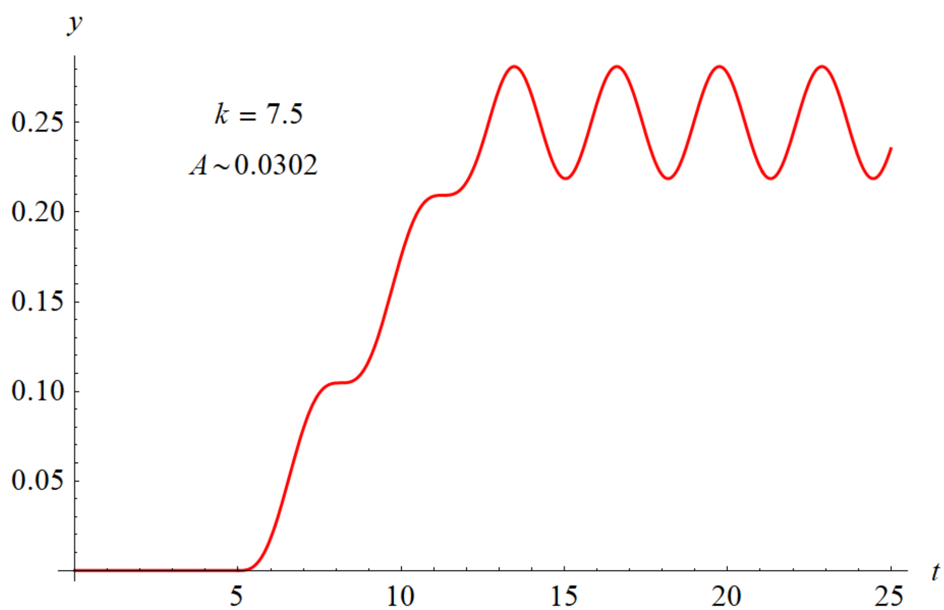

y
0.30
0.25
0.20
0.15
0.10
0.05
k = 5
A ~ 0.0468
5
10
15
20
t

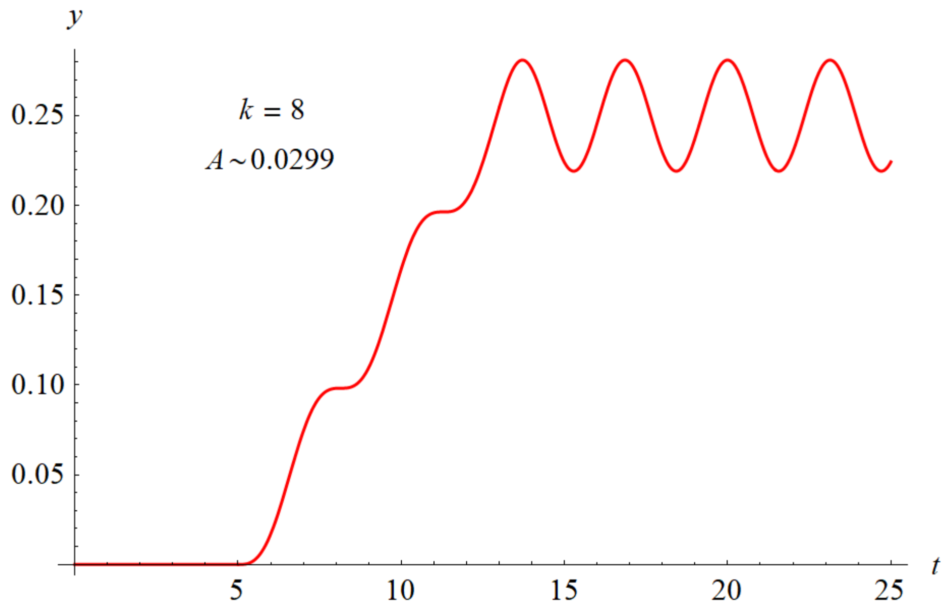

y
0.25
0.20
0.15
0.10
0.05
k = 5.5
A ~ 0.031
5
10
15
20
t

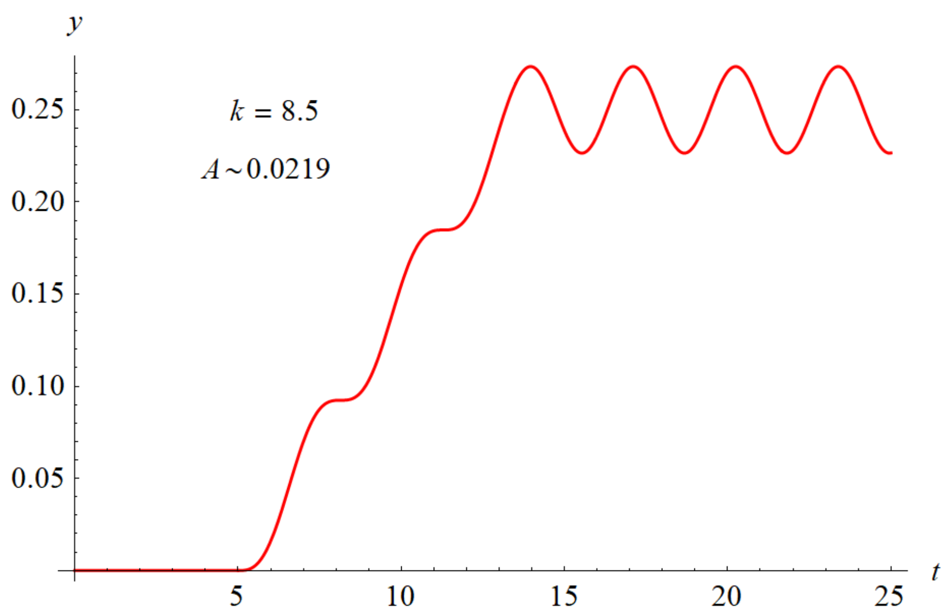

y
0.25
0.20
0.15
0.10
0.05
k = 6
A ~ 0.0101
5
10
15
20
t

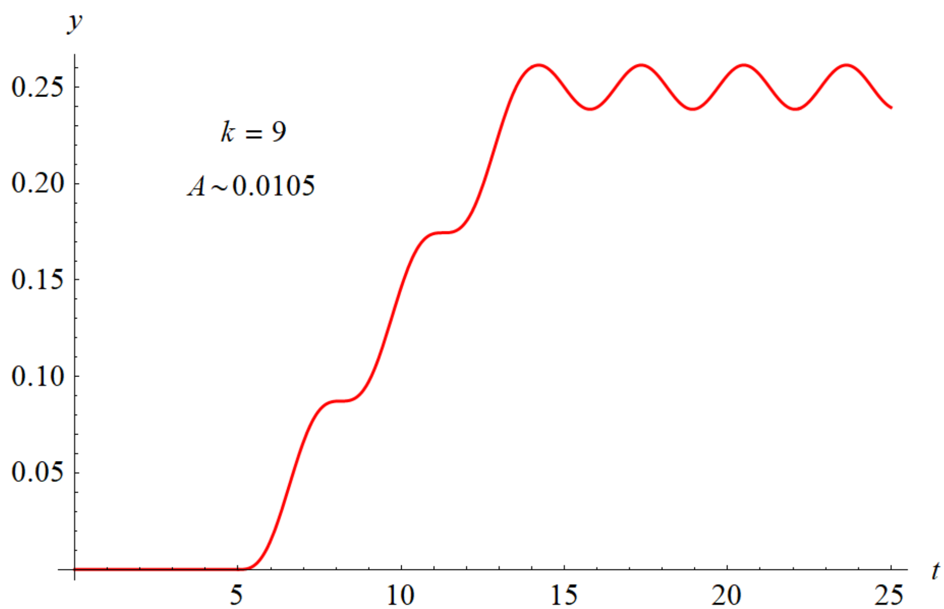

y
0.25
0.20
0.15
0.10
0.05
k = 6.5
A ~ 0.0073
5
10
15
20
25
t

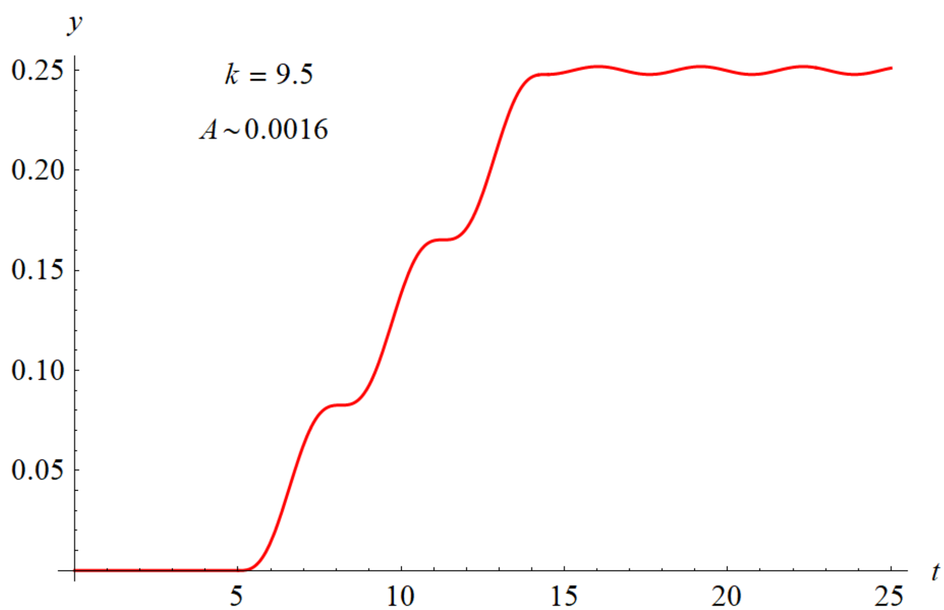

y
k = 7
A ~ 0.0219
0.25
0.20
0.15
0.10
0.05
5
10
15
20
25
t

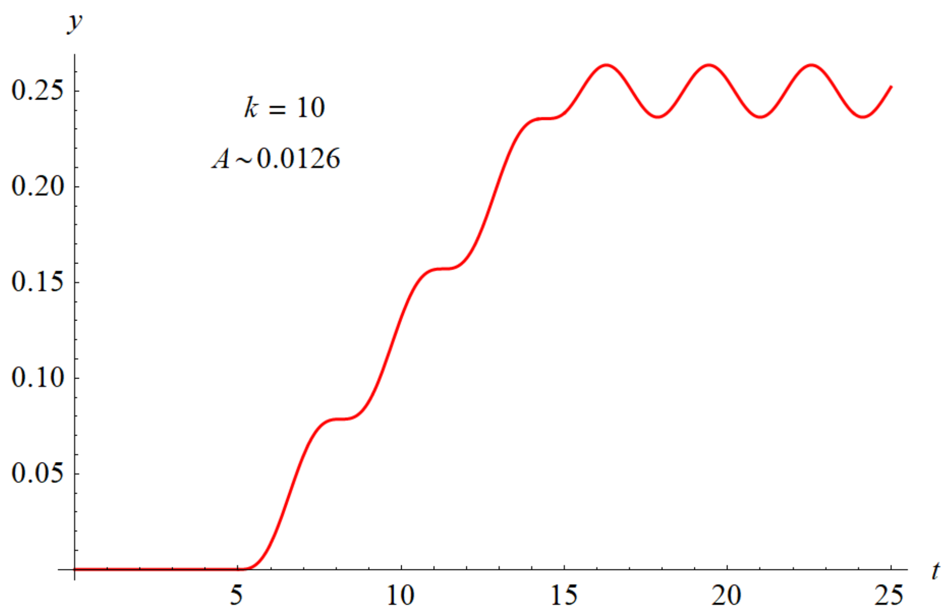

y
k = 7.5
A ~ 0.0302
0.25
0.20
0.15
0.10
0.05
5
10
15
20
25
t

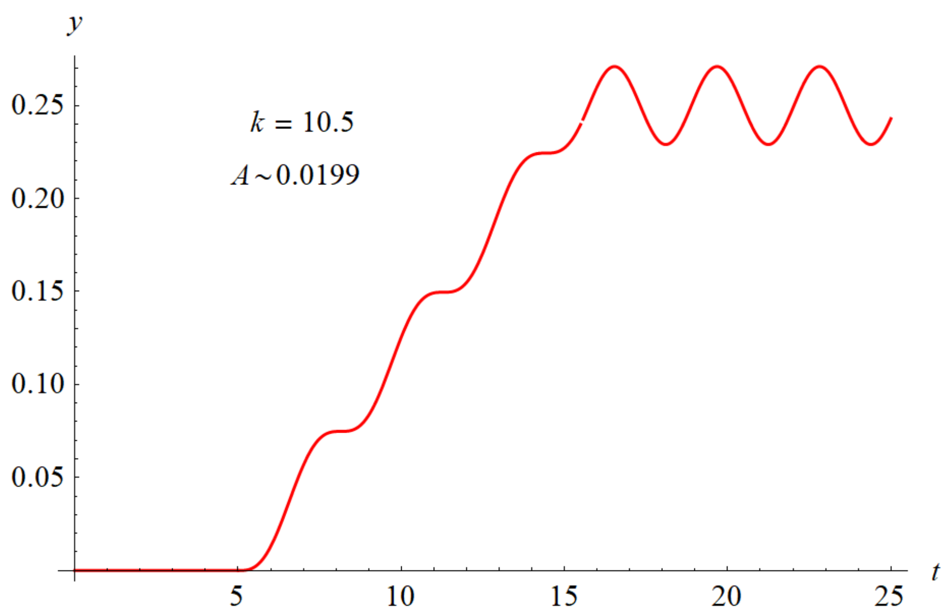

y
k = 8
A ~ 0.0299
0.25
0.20
0.15
0.10
0.05
5
10
15
20
25
t

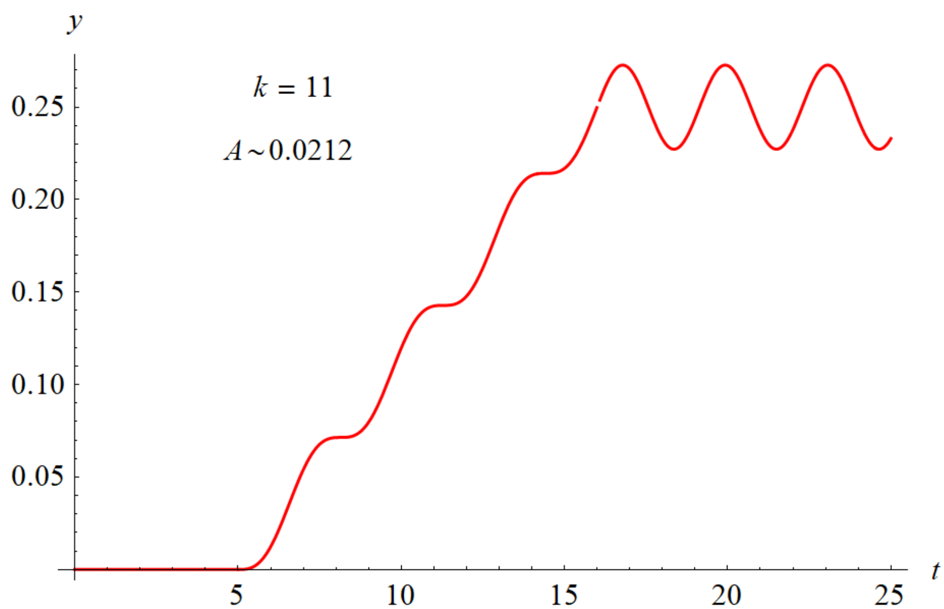

y
k = 8.5
A ~ 0.0219
0.25
0.20
0.15
0.10
0.05
5
10
15
20
25
t

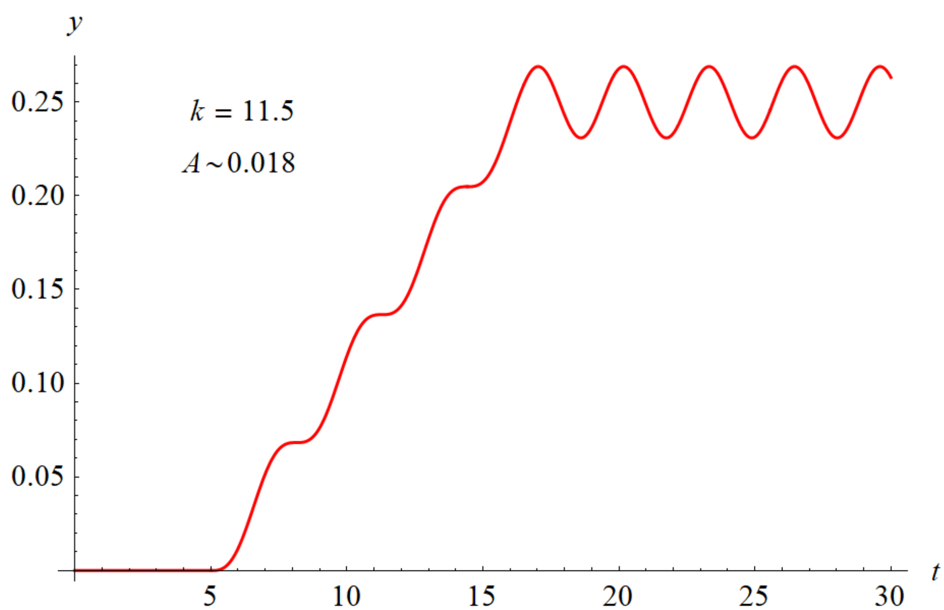

y
0.25
0.20
0.15
0.10
0.05
k = 9
A ~ 0.0105
5
10
15
20
25
t

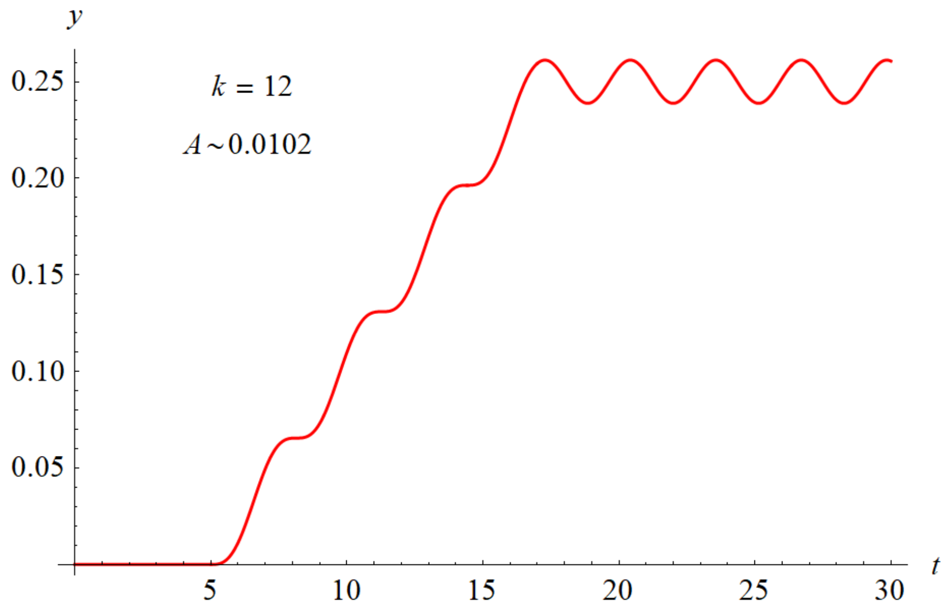

y
0.25
0.20
0.15
0.10
0.05
k = 9.5
A ~ 0.0016
5
10
15
20
25
t

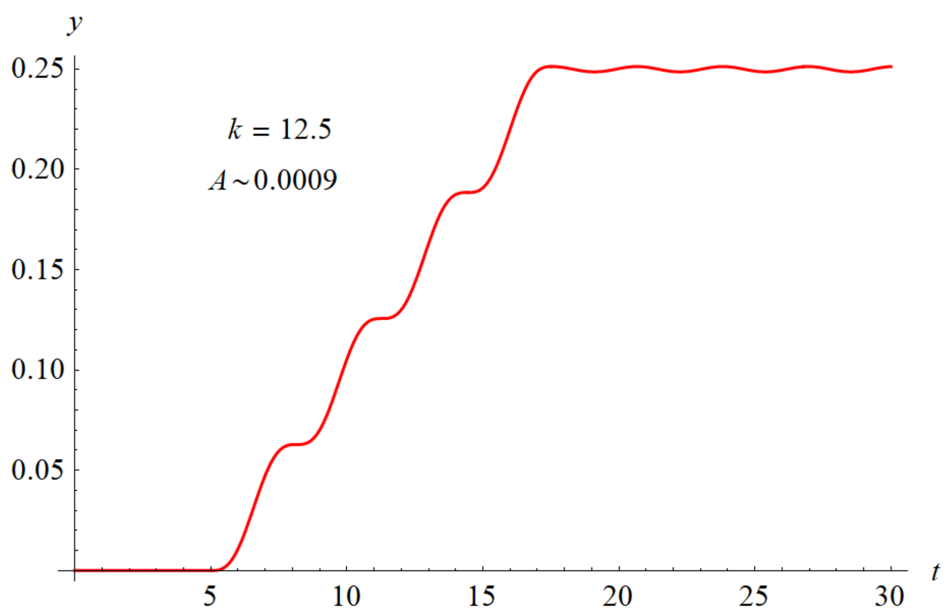

y
0.25
0.20
0.15
0.10
0.05
k = 10
A ~ 0.0126
5
10
15
20
25
t

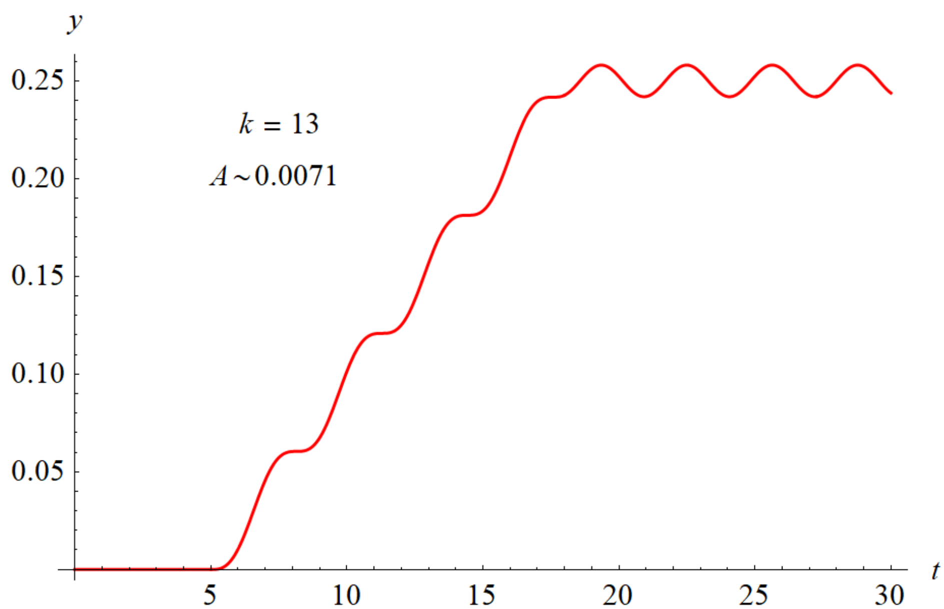

y
0.25
0.20
0.15
0.10
0.05
k = 10.5
A ~ 0.0199
5
10
15
20
25
t

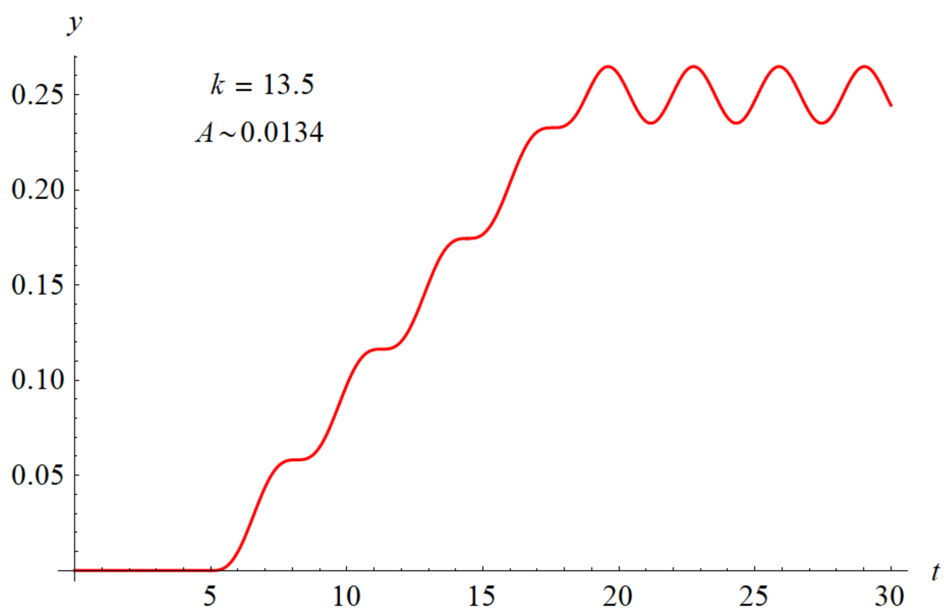

y
k = 11
A ~ 0.0212
0.25
0.20
0.15
0.10
0.05
5
10
15
20
25
t

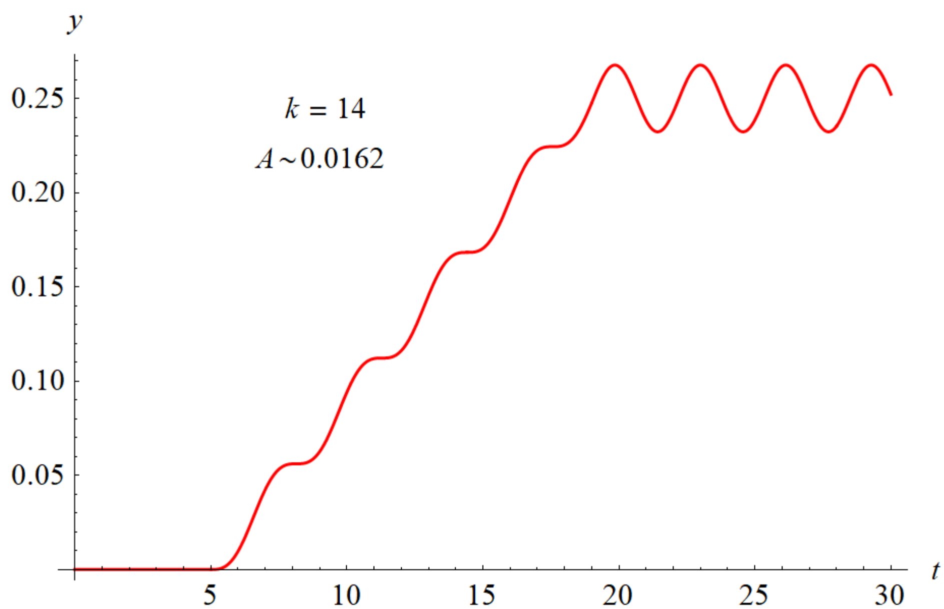

y
k = 11.5
A ~ 0.018
0.25
0.20
0.15
0.10
0.05
5
10
15
20
25
30
t

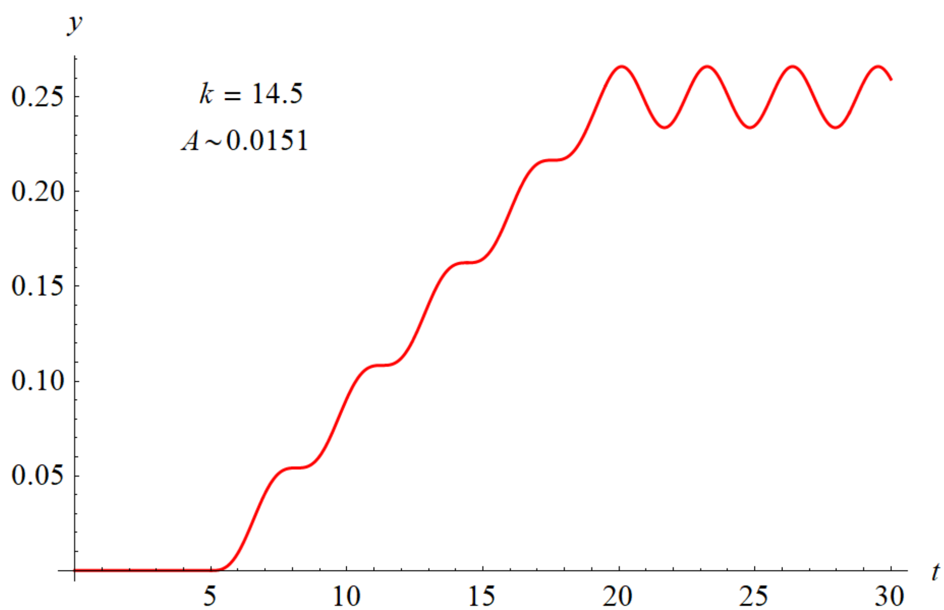

y
0.25
0.20
0.15
0.10
0.05
k = 12
A ~ 0.0102
5
10
15
20
25
30
t

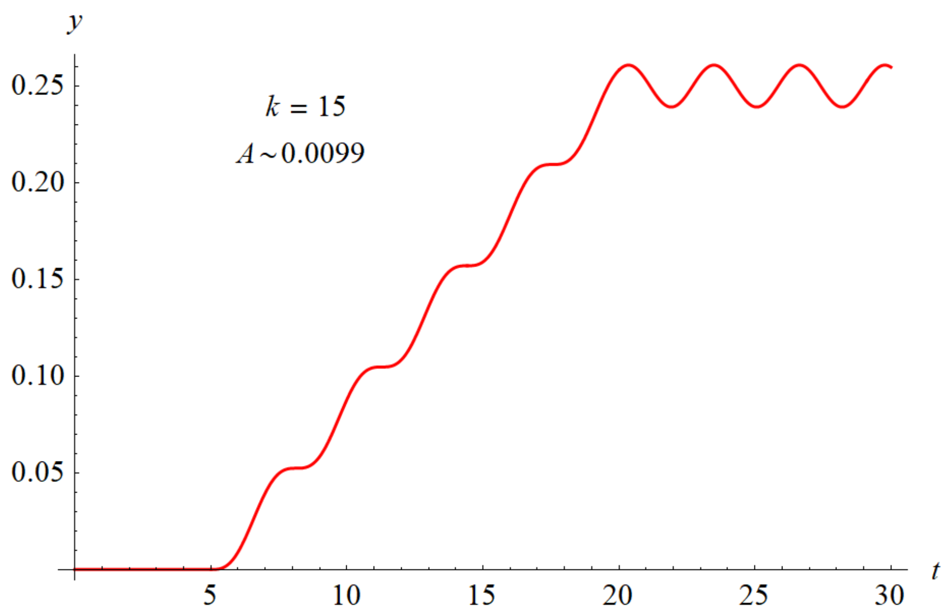

y
0.25
0.20
0.15
0.10
0.05
k = 12.5
A ~ 0.0009
5
10
15
20
25
30
t


y
0.25
0.20
0.15
0.10
0.05
k = 13
A ~ 0.0071
5
10
15
20
25
30
t


y
0.25
0.20
0.15
0.10
0.05
k = 13.5
A ~ 0.0134
5
10
15
20
25
30
t


y
k = 14
A ~ 0.0162
0.25
0.20
0.15
0.10
0.05
5
10
15
20
25
30
t


y
k = 14.5
A ~ 0.0151
0.25
0.20
0.15
0.10
0.05
5
10
15
20
25
30
t



Now plot the values of A versus the values of k to determine the functional dependence.