Problem 23

Consider the initial value problem

$$y'' + y = h(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{11k/4}(t).$$

Observe that this problem is identical to Problem 19 except that the frequency of the forcing term has been increased somewhat.

(a) Find the solution of this initial value problem.

(b) Let $n \geq 33$ and plot the solution for $0 \leq t \leq 90$ or longer. Your plot should show a clearly recognizable beat.

(c) From the graph in part (b), estimate the "slow period" and the "fast period" for this oscillator.

(d) For a sinusoidally forced oscillator, it was shown in Section 3.8 that the "slow frequency" is given by $|\omega - \omega_0|/2$, where ω_0 is the natural frequency of the system and ω is the forcing frequency. Similarly, the "fast frequency" is $(\omega + \omega_0)/2$. Use these expressions to calculate the "fast period" and the "slow period" for the oscillator in this problem. How well do the results compare with your estimates from part (c)?

[TYPO: $f(t)$ should be $h(t)$.]

Solution

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Substitute the provided function for $h(t)$ and take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{11k/4}(t)\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_0(t)\} + 2\sum_{k=1}^{n}(-1)^k \mathcal{L}\{u_{11k/4}(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \int_0^\infty e^{-st}u_0(t)\,dt + 2\sum_{k=1}^{n}(-1)^k \int_0^\infty e^{-st}u_{11k/4}(t)\,dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + Y(s) = \int_0^\infty e^{-st}\,dt + 2\sum_{k=1}^{n}(-1)^k \int_{11k/4}^\infty e^{-st}\,dt$$

$$(s^2+1)Y(s) = \frac{1}{s} + 2\sum_{k=1}^{n}(-1)^k\left(\frac{e^{-11ks/4}}{s}\right)$$

Solve for $Y(s)$.

$$Y(s) = \frac{1}{s(s^2+1)} + 2\sum_{k=1}^{n}(-1)^k\left[\frac{1}{s(s^2+1)}\right]e^{-11ks/4}$$

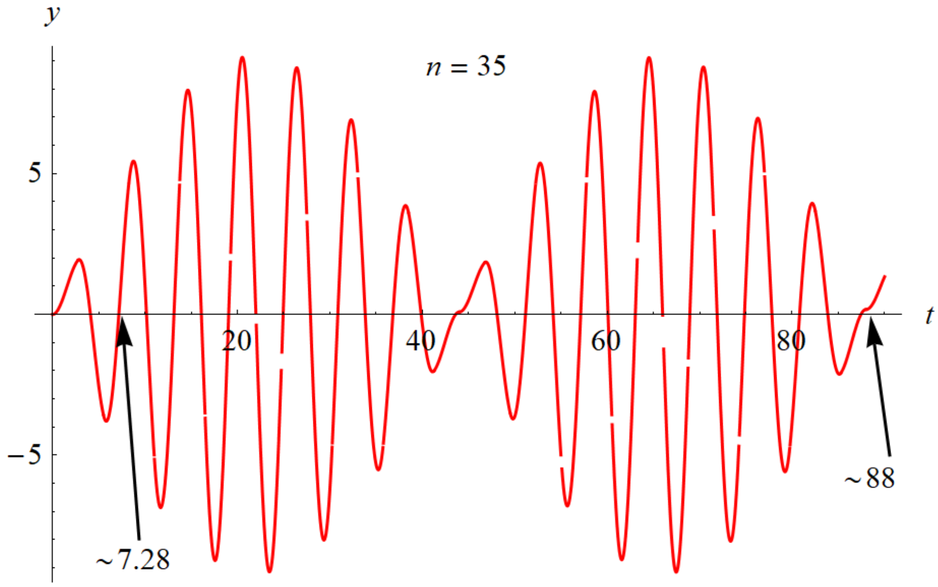
Now write it in terms of known transforms by using partial fraction decomposition.

$$Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 2\sum_{k=1}^{n}(-1)^k\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-11ks/4}$$

Take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 2\sum_{k=1}^{n}(-1)^k\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-11ks/4}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} + 2\sum_{k=1}^{n}(-1)^k\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-11ks/4}\right\} \\
&= 1 - \cos t + 2\sum_{k=1}^{n}(-1)^k[1 - \cos(t - 11k/4)]H(t - 11k/4) \\
&= 1 - \cos t + 2\sum_{k=1}^{n}(-1)^k[1 - \cos(t - 11k/4)]u_{11k/4}(t)
\end{aligned}$$

Below is a plot of $y(t)$ versus t for $n = 35$.



The slow period is about 88, and the fast period is about 7.28. As a result, the slow frequency is about $2\pi/88 = \pi/44 \approx 0.0714$, and the fast frequency is about $2\pi/7.28 = 25\pi/91 \approx 0.863$. The natural frequency of this system is $\omega_0^2 = 1$, where 1 is the coefficient of y in the ODE. On the other hand, the forcing period is $2.75 \times 2 = 5.5$, which means the forcing frequency is $\omega = 2\pi/5.5 = 4\pi/11$. Using the formulas, the slow and fast frequencies are

$$\text{Slow Frequency:} \quad \frac{|\omega - \omega_0|}{2} = \frac{\left|\frac{4\pi}{11} - 1\right|}{2} = \frac{4\pi - 11}{22} \approx 0.0712$$

$$\text{Fast Frequency:} \quad \frac{\omega + \omega_0}{2} = \frac{\frac{4\pi}{11} + 1}{2} = \frac{4\pi + 11}{22} \approx 1.07.$$

Now calculate the percent differences to see how far off these results are from the estimates.

$$\text{Slow Frequency:} \quad \frac{\frac{\pi}{44} - \frac{4\pi-11}{22}}{\frac{4\pi-11}{22}} \times 100\% \approx 0.283\%$$

$$\text{Fast Frequency:} \quad \frac{\frac{25\pi}{91} - \frac{4\pi+11}{22}}{\frac{4\pi+11}{22}} \times 100\% \approx -19.4\%$$

Therefore, the measured estimate for the slow frequency is 0.283% higher than the calculated value, and the measured estimate for the fast frequency is 19.4% less than the calculated value.