Problem 13

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y^{(4)} + 5y'' + 4y = 1 - u_\pi(t); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the derivatives transform as follows.

$$\begin{aligned}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)
\end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y^{(4)} + 5y'' + 4y\} = \mathcal{L}\{1 - u_\pi(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y^{(4)}\} + 5\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{u_\pi(t)\}$$

$$\begin{aligned}
[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] + 5[s^2Y(s) - sy(0) - y'(0)] \\
+ 4[Y(s)] = \int_0^\infty e^{-st}(1)\, dt - \int_0^\infty e^{-st}[u_\pi(t)]\, dt
\end{aligned}$$

Plug in the initial conditions, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$, and $y'''(0) = 0$.

$$[s^4Y(s)] + 5[s^2Y(s)] + 4[Y(s)] = \int_0^\infty e^{-st}\, dt - \int_\pi^\infty e^{-st}\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$\begin{aligned}
(s^4 + 5s^2 + 4)Y(s) &= \left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \left(-\frac{1}{s}e^{-st}\right)\Big|_\pi^\infty \\
&= \frac{1}{s} - \frac{1}{s}e^{-\pi s}
\end{aligned}$$

Solve for $Y(s)$ and write the right side in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{1}{s(s^4+5s^2+4)} - \frac{1}{s(s^4+5s^2+4)}e^{-\pi s} \\ &= \frac{1}{s(s^2+1)(s^2+4)} - \frac{1}{s(s^2+1)(s^2+4)}e^{-\pi s} \end{aligned}$$

Use partial fraction decomposition.

$$\frac{1}{s(s^2+1)(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{s^2+4}$$

Multiply both sides by $s(s^2+1)(s^2+4)$.

$$1 = A(s^2+1)(s^2+4) + (Bs+C)s(s^2+4) + (Ds+E)s(s^2+1)$$

Plug in five random values for s to obtain a system of five equations for A, B, C, D, and E.

$$\begin{aligned} s=0:&\quad 1 = 4A \\ s=1:&\quad 1 = 10A+5B+5C+2D+2E \\ s=2:&\quad 1 = 40A+32B+16C+20D+10E \\ s=3:&\quad 1 = 130A+117B+39C+90D+30E \\ s=4:&\quad 1 = 340A+320B+80C+272D+68E \end{aligned}$$

Solving this system yields $A = 1/4$, $B = -1/3$, $C = 0$, $D = 1/12$, and $E = 0$.

$$Y(s) = \left(\frac{1/4}{s} + \frac{-\frac{1}{3}s}{s^2+1} + \frac{\frac{1}{12}s}{s^2+4}\right) - \left(\frac{1/4}{s} + \frac{-\frac{1}{3}s}{s^2+1} + \frac{\frac{1}{12}s}{s^2+4}\right)e^{-\pi s}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{1/4}{s} + \frac{-\frac{1}{3}s}{s^2+1} + \frac{\frac{1}{12}s}{s^2+4}\right) - \left(\frac{1/4}{s} + \frac{-\frac{1}{3}s}{s^2+1} + \frac{\frac{1}{12}s}{s^2+4}\right)e^{-\pi s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1/4}{s} + \frac{-\frac{1}{3}s}{s^2+1} + \frac{\frac{1}{12}s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\left(\frac{1/4}{s} + \frac{-\frac{1}{3}s}{s^2+1} + \frac{\frac{1}{12}s}{s^2+4}\right)e^{-\pi s}\right\} \\ &= \frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t - \left[\frac{1}{4} - \frac{1}{3}\cos(t-\pi) + \frac{1}{12}\cos 2(t-\pi)\right]H(t-\pi) \\ &= \frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t - \left(\frac{1}{4} + \frac{1}{3}\cos t + \frac{1}{12}\cos 2t\right)H(t-\pi) \\ &= \frac{1}{12}(3 - 4\cos t + \cos 2t) - \frac{1}{12}(3 + 4\cos t + \cos 2t)H(t-\pi) \\ &= \frac{1}{12}\left(8\sin^4\frac{t}{2}\right) - \frac{1}{12}\left(8\cos^4\frac{t}{2}\right)H(t-\pi) \\ &= \frac{2}{3}\left[\sin^4\frac{t}{2} - H(t-\pi)\cos^4\frac{t}{2}\right] \\ &= \frac{2}{3}\left[\sin^4\frac{t}{2} - u_\pi(t)\cos^4\frac{t}{2}\right] \end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $f(t)$ versus t.