# Problem 16

A certain spring-mass system satisfies the initial value problem

$$u'' + \frac{1}{4}u' + u = kg(t), \qquad u(0) = 0, \quad u'(0) = 0,$$

where $g(t) = u_{3/2}(t) - u_{5/2}(t)$ and $k > 0$ is a parameter.

(a) Sketch the graph of $g(t)$. Observe that it is a pulse of unit magnitude extending over one time unit.

(b) Solve the initial value problem.

(c) Plot the solution for $k = 1/2$, $k = 1$, and $k = 2$. Describe the principal features of the solution and how they depend on $k$.

(d) Find, to two decimal places, the smallest value of $k$ for which the solution $u(t)$ reaches the value 2.

(e) Suppose $k = 2$. Find the time $\tau$ after which $|u(t)| < 0.1$ for all $t > \tau$.

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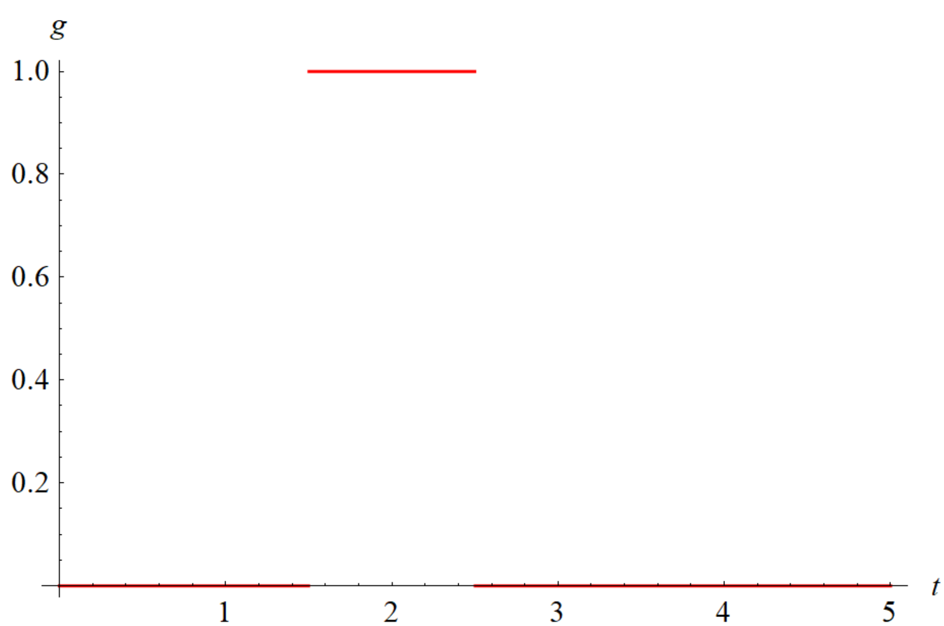
**Solution**

Here is a plot of $g(t)$ versus $t$.




Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $u(t)$ is defined here as

$$U(s) = \mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} u(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{du}{dt}\right\} = sU(s) - u(0)$$
$$\mathcal{L}\left\{\frac{d^2u}{dt^2}\right\} = s^2U(s) - su(0) - u'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\left\{u'' + \frac{1}{4}u' + u\right\} = \mathcal{L}\{kg(t)\} = \mathcal{L}\{k[u_{3/2}(t) - u_{5/2}(t)]\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{u''\} + \frac{1}{4}\mathcal{L}\{u'\} + \mathcal{L}\{u\} = k\mathcal{L}\{u_{3/2}(t)\} - k\mathcal{L}\{u_{5/2}(t)\}$$

$$[s^2U(s) - su(0) - u'(0)] + \frac{1}{4}[sU(s) - u(0)] + [U(s)] = k\int_0^\infty e^{-st}[u_{3/2}(t)]\,dt - k\int_0^\infty e^{-st}[u_{5/2}(t)]\,dt$$

Plug in the initial conditions, $u(0) = 0$ and $u'(0) = 0$.

$$[s^2U(s)] + \frac{1}{4}[sU(s)] + [U(s)] = k\int_{3/2}^\infty e^{-st}\,dt - k\int_{5/2}^\infty e^{-st}\,dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for $U$, the transformed solution.

$$\begin{aligned}\left(s^2 + \frac{1}{4}s + 1\right)U(s) &= k\left(-\frac{1}{s}e^{-st}\right)\bigg|_{3/2}^\infty - k\left(-\frac{1}{s}e^{-st}\right)\bigg|_{5/2}^\infty \\ &= k\left(\frac{1}{s}e^{-3s/2}\right) - k\left(\frac{1}{s}e^{-5s/2}\right)\end{aligned}$$

Solve for $U(s)$ and write the right side in terms of known transforms.

$$U(s) = \frac{k}{s\left(s^2 + \frac{1}{4}s + 1\right)}e^{-3s/2} - \frac{k}{s\left(s^2 + \frac{1}{4}s + 1\right)}e^{-5s/2}$$

Use partial fraction decomposition.

$$\frac{k}{s\left(s^2 + \frac{1}{4}s + 1\right)} = \frac{A}{s} + \frac{Bs + C}{s^2 + \frac{1}{4}s + 1}$$

Multiply both sides by $s\left(s^2 + \frac{1}{4}s + 1\right)$.

$$k = A\left(s^2 + \frac{1}{4}s + 1\right) + (Bs + C)s$$

Plug in three random values for $s$ to get a system of three equations for $A$, $B$, and $C$.

$$\begin{aligned} s = 0: &\quad k = A \\ s = 1: &\quad k = \frac{9}{4}A + B + C \\ s = 2: &\quad k = \frac{11}{2}A + 4B + 2C \end{aligned}$$

Solving this system yields $A = k$, $B = -k$, and $C = -k/4$.

$$U(s) = \left(\frac{k}{s} + \frac{-ks - \frac{k}{4}}{s^2 + \frac{1}{4}s + 1}\right)e^{-3s/2} - \left(\frac{k}{s} + \frac{-ks - \frac{k}{4}}{s^2 + \frac{1}{4}s + 1}\right)e^{-5s/2}$$

Complete the square in the denominators.

$$\begin{aligned} U(s) &= \left(\frac{k}{s} + \frac{-ks - \frac{k}{4}}{s^2 + \frac{1}{4}s + \frac{1}{64} + 1 - \frac{1}{64}}\right)e^{-3s/2} - \left(\frac{k}{s} + \frac{-ks - \frac{k}{4}}{s^2 + \frac{1}{4}s + \frac{1}{64} + 1 - \frac{1}{64}}\right)e^{-5s/2} \\ &= \left[\frac{k}{s} - k\frac{s + \frac{1}{4}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}}\right]e^{-3s/2} - \left[\frac{k}{s} - k\frac{s + \frac{1}{4}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}}\right]e^{-5s/2} \end{aligned}$$

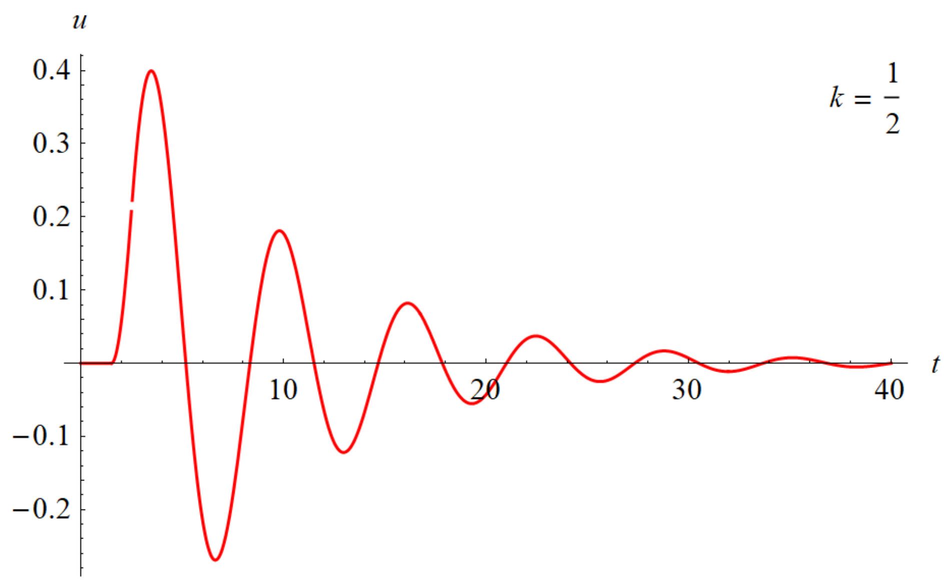
Make it so that $s + \frac{1}{8}$ appears in the numerators.

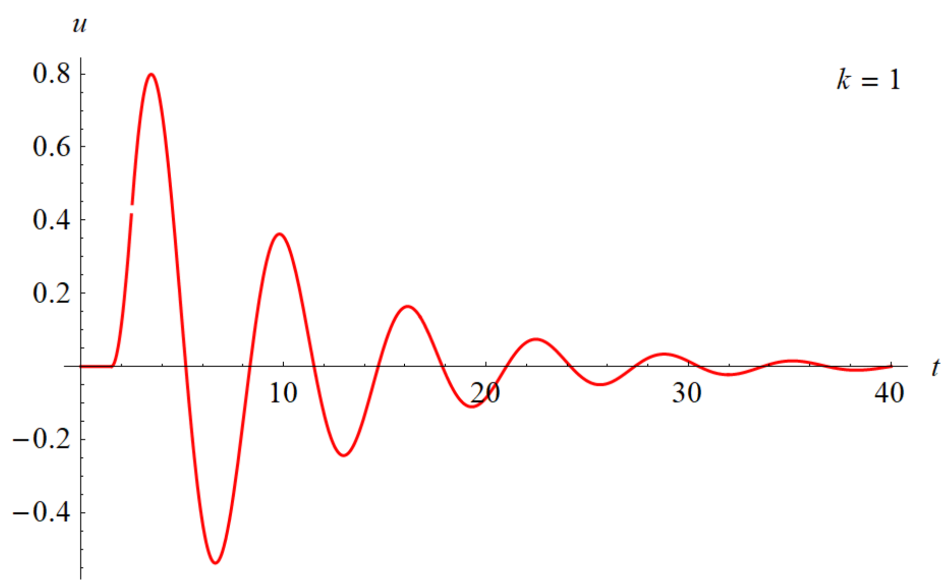
$$\begin{aligned} U(s) &= \left[\frac{k}{s} - k\frac{\left(s + \frac{1}{8}\right) + \frac{1}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}}\right]e^{-3s/2} - \left[\frac{k}{s} - k\frac{\left(s + \frac{1}{8}\right) + \frac{1}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}}\right]e^{-5s/2} \\ &= \left[\frac{k}{s} - k\frac{s + \frac{1}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}} - k\frac{\frac{1}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}}\right]e^{-3s/2} - \left[\frac{k}{s} - k\frac{s + \frac{1}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}} - k\frac{\frac{1}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}}\right]e^{-5s/2} \\ &= \left[\frac{k}{s} - k\frac{s + \frac{1}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}} - \frac{k}{\sqrt{63}}\frac{\frac{\sqrt{63}}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}}\right]e^{-3s/2} - \left[\frac{k}{s} - k\frac{s + \frac{1}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}} - \frac{k}{\sqrt{63}}\frac{\frac{\sqrt{63}}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}}\right]e^{-5s/2} \end{aligned}$$

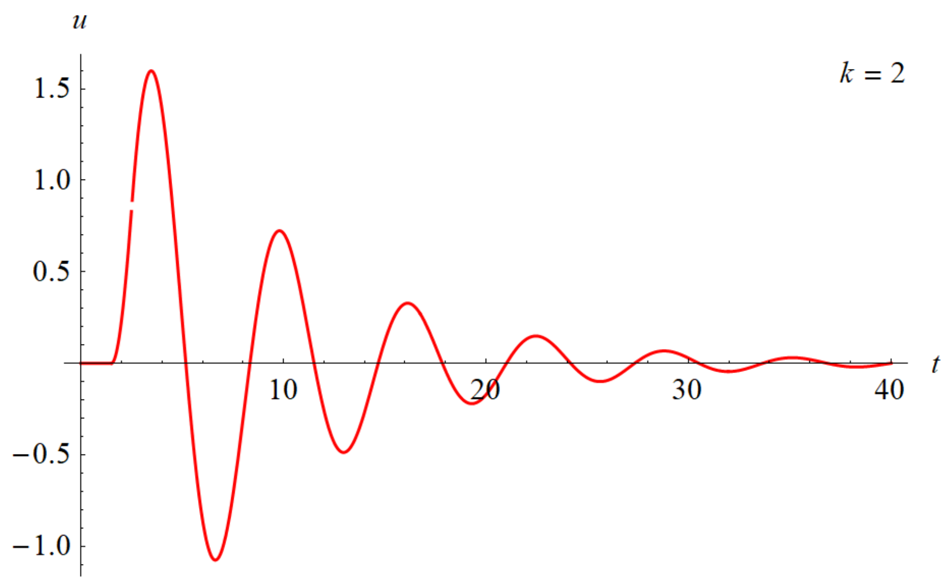
Now take the inverse Laplace transform of $U(s)$ to get $u(t)$.

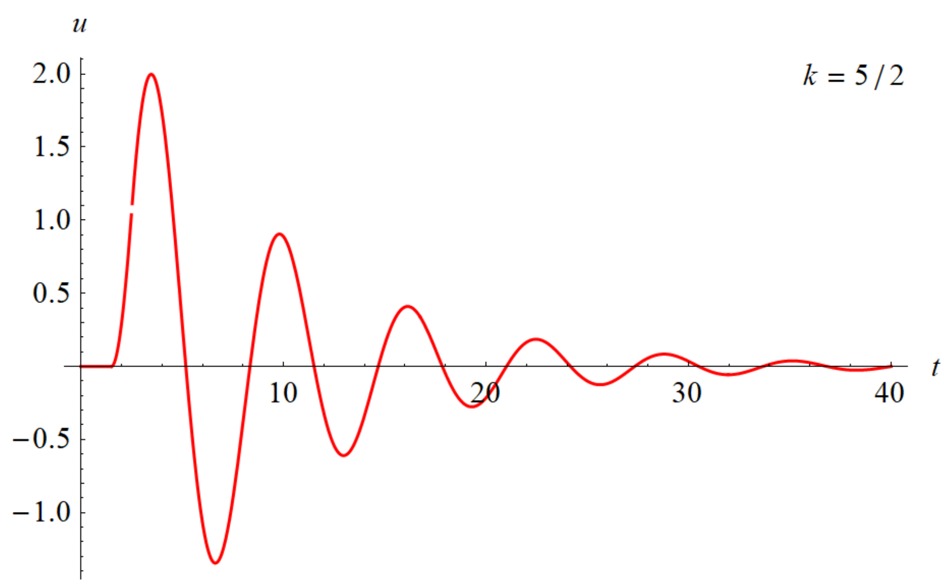
$$\begin{aligned} u(t) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \mathcal{L}^{-1}\left\{\left[\frac{k}{s} - k\frac{s + \frac{1}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}} - \frac{k}{\sqrt{63}}\frac{\frac{\sqrt{63}}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}}\right]e^{-3s/2}\right\} \\ &\qquad - \mathcal{L}^{-1}\left\{\left[\frac{k}{s} - k\frac{s + \frac{1}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}} - \frac{k}{\sqrt{63}}\frac{\frac{\sqrt{63}}{8}}{\left(s + \frac{1}{8}\right)^2 + \frac{63}{64}}\right]e^{-5s/2}\right\} \\ &= \left\{k - ke^{-\left(t - \frac{3}{2}\right)/8}\cos\left[\frac{\sqrt{63}}{8}\left(t - \frac{3}{2}\right)\right] - \frac{k}{\sqrt{63}}e^{-\left(t - \frac{3}{2}\right)/8}\sin\left[\frac{\sqrt{63}}{8}\left(t - \frac{3}{2}\right)\right]\right\}H\left(t - \frac{3}{2}\right) \\ &\qquad - \left\{k - ke^{-\left(t - \frac{5}{2}\right)/8}\cos\left[\frac{\sqrt{63}}{8}\left(t - \frac{5}{2}\right)\right] - \frac{k}{\sqrt{63}}e^{-\left(t - \frac{5}{2}\right)/8}\sin\left[\frac{\sqrt{63}}{8}\left(t - \frac{5}{2}\right)\right]\right\}H\left(t - \frac{5}{2}\right) \end{aligned}$$

Below are several plots of $u(t)$ versus $t$ for several values of $k$.


u
0.4
0.3
0.2
0.1
−0.1
−0.2
10
20
30
40
t
k = 1/2


u
0.8
0.6
0.4
0.2
−0.2
−0.4
10
20
30
40
t
k = 1




The first maximum of $u$ seems to be proportional to $k$; the maximum is 2 when $k \approx 2.51$. The solution starts to oscillate at $t = 3/2$ and has an amplitude that decays with time.

Adjust the lower and upper bounds of the $k = 2$ graph to $-0.1$ and $0.1$, respectively, to find the time at which the amplitude is smaller than 0.1 for all later times.