# Problem 18

Consider the initial value problem

$$y'' + \frac{1}{3}y' + 4y = f_k(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f_k(t) = \begin{cases} 1/2k, & 4 - k \le t < 4 + k \\ 0, & 0 \le t < 4 - k \quad \text{and} \quad t \ge 4 + k \end{cases}$$

and $0 < k < 4$.

(a) Sketch the graph of $f_k(t)$. Observe that the area under the graph is independent of $k$. If $f_k(t)$ represents a force, this means that the product of the magnitude of the force and the time interval during which it acts does not depend on $k$.

(b) Write $f_k(t)$ in terms of the unit step function and then solve the given initial value problem.

(c) Plot the solution for $k = 2$, $k = 1$, and $k = \frac{1}{2}$. Describe how the solution depends on $k$.

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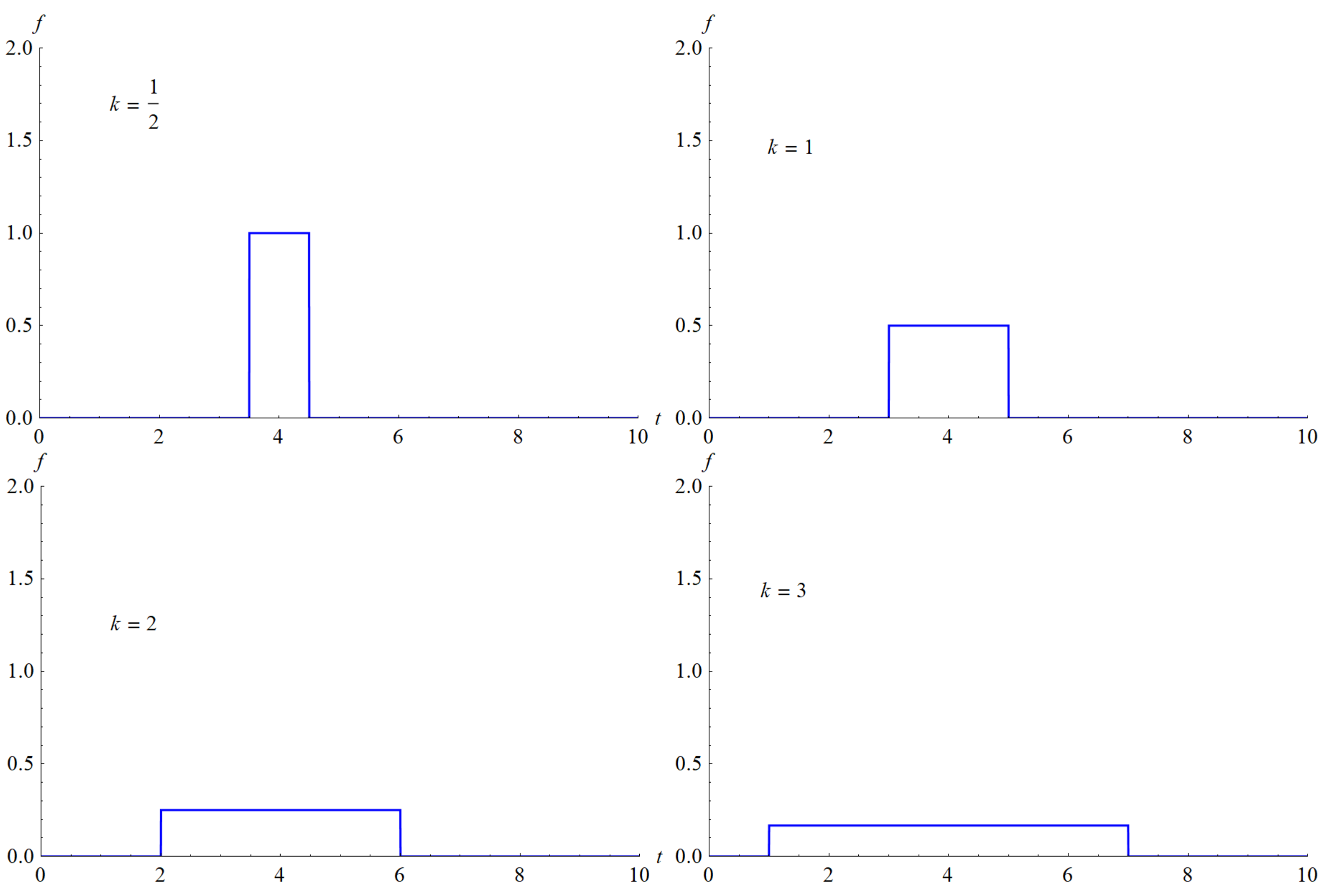
## Solution

### Part (a)



## Part (b)

To make the function have the value $1/2k$ at $t = 4 - k$, use the Heaviside function $H[t - (4 - k)]$, which is defined to be 1 if $t > 4 - k$ and 0 if $t < 4 - k$. A second term is needed to cancel this one once $t > 4 + k$.

$$\begin{aligned} f_k(t) &= \frac{1}{2k}H[t-(4-k)] - \frac{1}{2k}H[t-(4+k)] \\ &= \frac{1}{2k}\{H[t-(4-k)] - H[t-(4+k)]\} \\ &= \frac{1}{2k}[u_{4-k}(t) - u_{4+k}(t)] \end{aligned}$$

The initial value problem to solve then is

$$y'' + \frac{1}{3}y' + 4y = \frac{1}{2k}[u_{4-k}(t) - u_{4+k}(t)], \qquad y(0) = 0, \quad y'(0) = 0.$$

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\left\{y'' + \frac{1}{3}y' + 4y\right\} = \mathcal{L}\left\{\frac{1}{2k}[u_{4-k}(t) - u_{4+k}(t)]\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \frac{1}{3}\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \frac{1}{2k}\mathcal{L}\{u_{4-k}(t)\} - \frac{1}{2k}\mathcal{L}\{u_{4+k}(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + \frac{1}{3}[sY(s) - y(0)] + 4[Y(s)] = \frac{1}{2k}\int_0^\infty e^{-st}[u_{4-k}(t)]\, dt - \frac{1}{2k}\int_0^\infty e^{-st}[u_{4+k}(t)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + \frac{1}{3}[sY(s)] + 4[Y(s)] = \frac{1}{2k}\int_{4-k}^\infty e^{-st}\, dt - \frac{1}{2k}\int_{4+k}^\infty e^{-st}\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for $Y$, the transformed solution.

$$\left(s^2 + \frac{1}{3}s + 4\right)Y(s) = \frac{1}{2k}\left[\frac{e^{-(4-k)s}}{s}\right] - \frac{1}{2k}\left[\frac{e^{-(4+k)s}}{s}\right]$$

Solve for $Y(s)$ and write the right side in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{1}{2k}\frac{1}{s\left(s^2+\frac{1}{3}s+4\right)}e^{-(4-k)s} - \frac{1}{2k}\frac{1}{s\left(s^2+\frac{1}{3}s+4\right)}e^{-(4+k)s} \\ &= \frac{1}{2k}\left(\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s-\frac{1}{12}}{s^2+\frac{1}{3}s+4}\right)e^{-(4-k)s} - \frac{1}{2k}\left(\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s-\frac{1}{12}}{s^2+\frac{1}{3}s+4}\right)e^{-(4+k)s} \end{aligned}$$

Complete the square in the denominators.

$$\begin{aligned} Y(s) &= \frac{1}{2k}\left(\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s-\frac{1}{12}}{s^2+\frac{1}{3}s+\frac{1}{36}+4-\frac{1}{36}}\right)e^{-(4-k)s} - \frac{1}{2k}\left(\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s-\frac{1}{12}}{s^2+\frac{1}{3}s+\frac{1}{36}+4-\frac{1}{36}}\right)e^{-(4+k)s} \\ &= \frac{1}{2k}\left[\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s-\frac{1}{12}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4-k)s} - \frac{1}{2k}\left[\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s-\frac{1}{12}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4+k)s} \end{aligned}$$

Make it so that $s+\frac{1}{6}$ appears in the numerators.

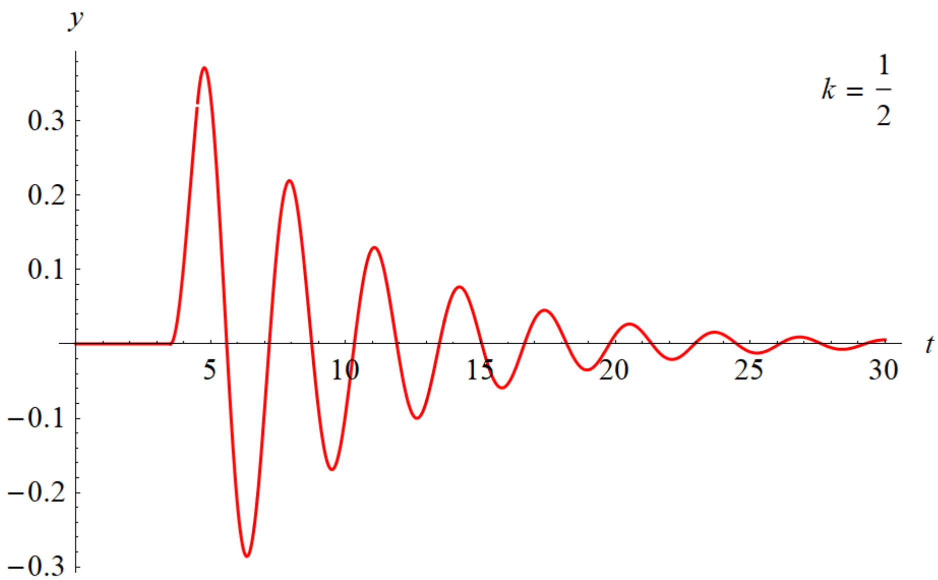
$$\begin{aligned} Y(s) &= \frac{1}{2k}\left[\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}\left(s+\frac{1}{6}\right)+\frac{1}{24}-\frac{1}{12}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4-k)s} - \frac{1}{2k}\left[\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}\left(s+\frac{1}{6}\right)+\frac{1}{24}-\frac{1}{12}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4+k)s} \\ &= \frac{1}{2k}\left[\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}\left(s+\frac{1}{6}\right)-\frac{1}{24}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4-k)s} - \frac{1}{2k}\left[\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}\left(s+\frac{1}{6}\right)-\frac{1}{24}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4+k)s} \\ &= \frac{1}{2k}\left[\frac{\frac{1}{4}}{s} - \frac{1}{4}\frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{1}{24}\frac{1}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4-k)s} \\ &\qquad - \frac{1}{2k}\left[\frac{\frac{1}{4}}{s} - \frac{1}{4}\frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{1}{24}\frac{1}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4+k)s} \\ &= \frac{1}{2k}\left[\frac{\frac{1}{4}}{s} - \frac{1}{4}\frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{1}{4\sqrt{143}}\frac{\frac{\sqrt{143}}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4-k)s} \\ &\qquad - \frac{1}{2k}\left[\frac{\frac{1}{4}}{s} - \frac{1}{4}\frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{1}{4\sqrt{143}}\frac{\frac{\sqrt{143}}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4+k)s} \end{aligned}$$

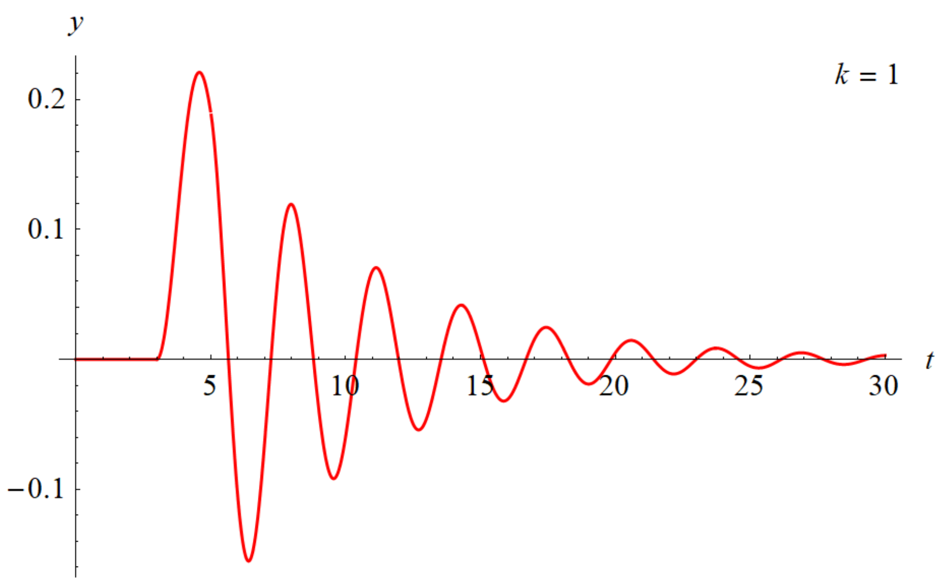
Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

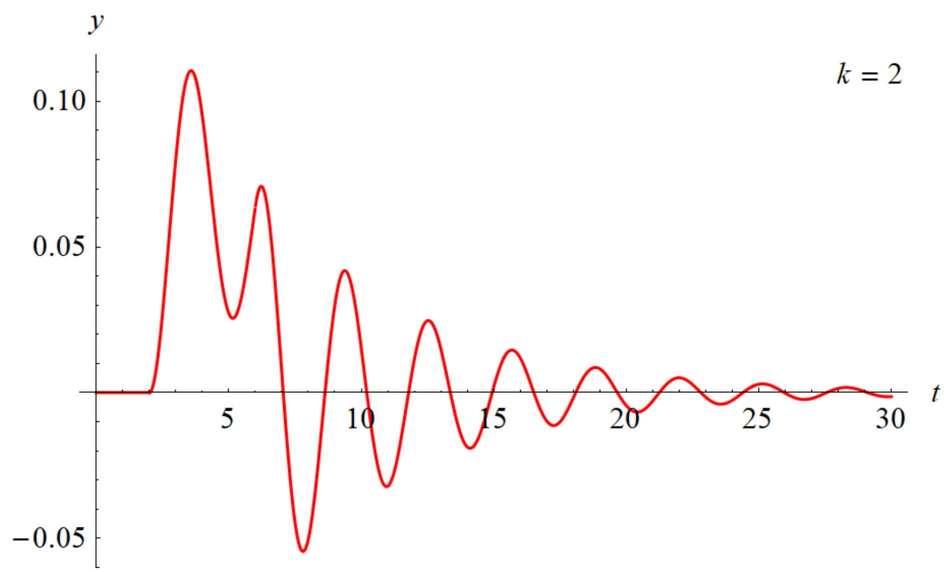
$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{2k}\left[\frac{\frac{1}{4}}{s} - \frac{1}{4}\frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{1}{4\sqrt{143}}\frac{\frac{\sqrt{143}}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4-k)s}\right. \\
&\qquad\qquad \left. -\frac{1}{2k}\left[\frac{\frac{1}{4}}{s} - \frac{1}{4}\frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{1}{4\sqrt{143}}\frac{\frac{\sqrt{143}}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4+k)s}\right\} \\
&= \frac{1}{2k}\mathcal{L}^{-1}\left\{\left[\frac{\frac{1}{4}}{s} - \frac{1}{4}\frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{1}{4\sqrt{143}}\frac{\frac{\sqrt{143}}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4-k)s}\right\} \\
&\qquad\qquad -\frac{1}{2k}\mathcal{L}^{-1}\left\{\left[\frac{\frac{1}{4}}{s} - \frac{1}{4}\frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{1}{4\sqrt{143}}\frac{\frac{\sqrt{143}}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right]e^{-(4+k)s}\right\} \\
&= \frac{1}{2k}\left\{\frac{1}{4} - \frac{1}{4}e^{-[t-(4-k)]/6}\cos\left[\frac{\sqrt{143}}{6}[t-(4-k)]\right] - \frac{1}{4\sqrt{143}}e^{-[t-(4-k)]/6}\sin\left[\frac{\sqrt{143}}{6}[t-(4-k)]\right]\right\}H[t-(4-k)] \\
&\qquad\qquad -\frac{1}{2k}\left\{\frac{1}{4} - \frac{1}{4}e^{-[t-(4+k)]/6}\cos\left[\frac{\sqrt{143}}{6}[t-(4+k)]\right] - \frac{1}{4\sqrt{143}}e^{-[t-(4+k)]/6}\sin\left[\frac{\sqrt{143}}{6}[t-(4+k)]\right]\right\}H[t-(4+k)] \\
&= \frac{1}{8k}\left\{1 - e^{-(t-4+k)/6}\cos\left[\frac{\sqrt{143}}{6}(t-4+k)\right] - \frac{1}{\sqrt{143}}e^{-(t-4+k)/6}\sin\left[\frac{\sqrt{143}}{6}(t-4+k)\right]\right\}u_{4-k}(t) \\
&\qquad\qquad -\frac{1}{8k}\left\{1 - e^{-(t-4-k)/6}\cos\left[\frac{\sqrt{143}}{6}(t-4-k)\right] - \frac{1}{\sqrt{143}}e^{-(t-4-k)/6}\sin\left[\frac{\sqrt{143}}{6}(t-4-k)\right]\right\}u_{4+k}(t)
\end{aligned}$$

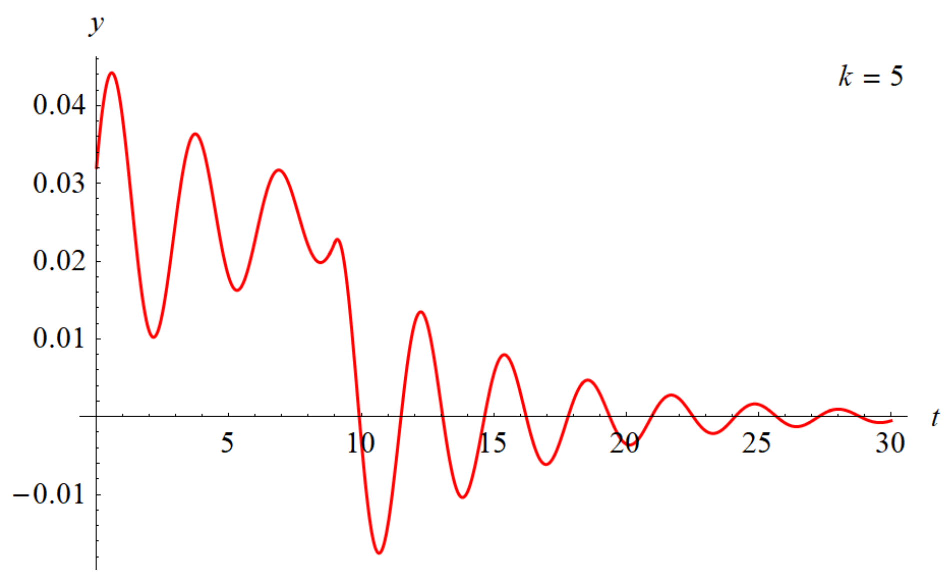
**Part (c)**

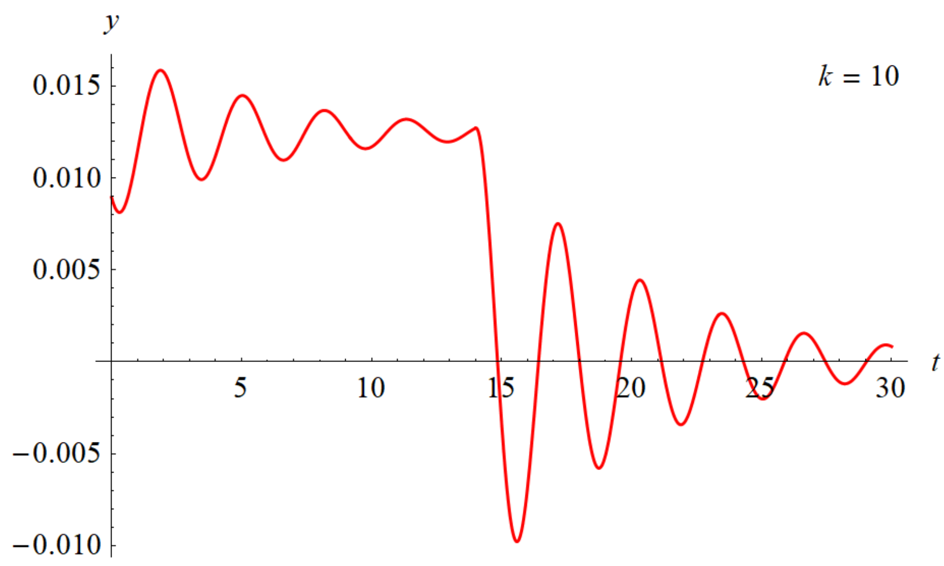
Below are plots for $y(t)$ versus $t$ for $k = 1/2$, $k = 1$, and $k = 2$.

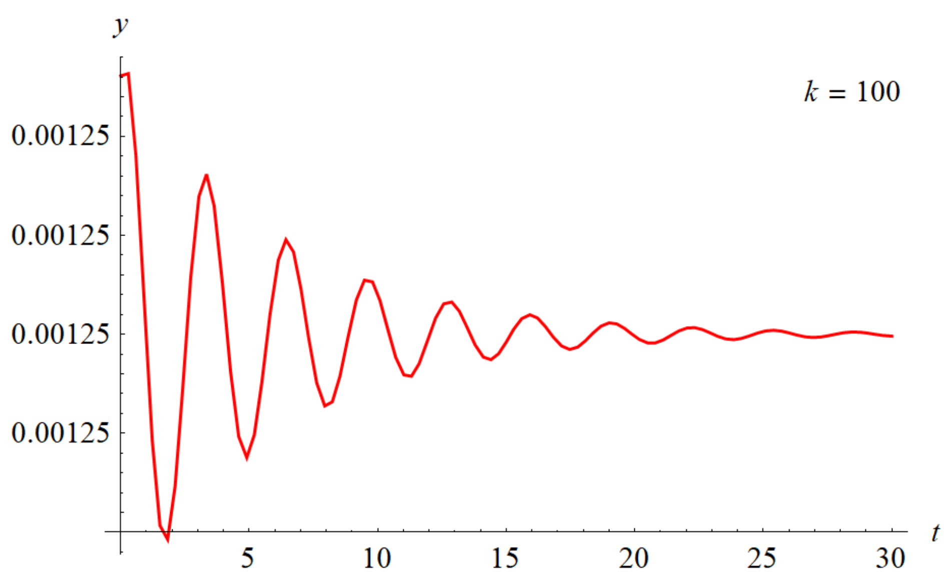



y
k = 2
0.10
0.05
−0.05
5
10
15
20
25
30
t


y
k = 5
0.04
0.03
0.02
0.01
−0.01
5
10
15
20
25
30
t




As $k$ increases, the amplitude of $y(t)$ decreases. Initially the graph gets closer to the $y$-axis, and then eventually rises above the $t$-axis.