Problem 19

Consider the initial value problem

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $f(t)$ on an interval such as $0 \le t \le 6\pi$.

(b) Find the solution of the initial value problem.

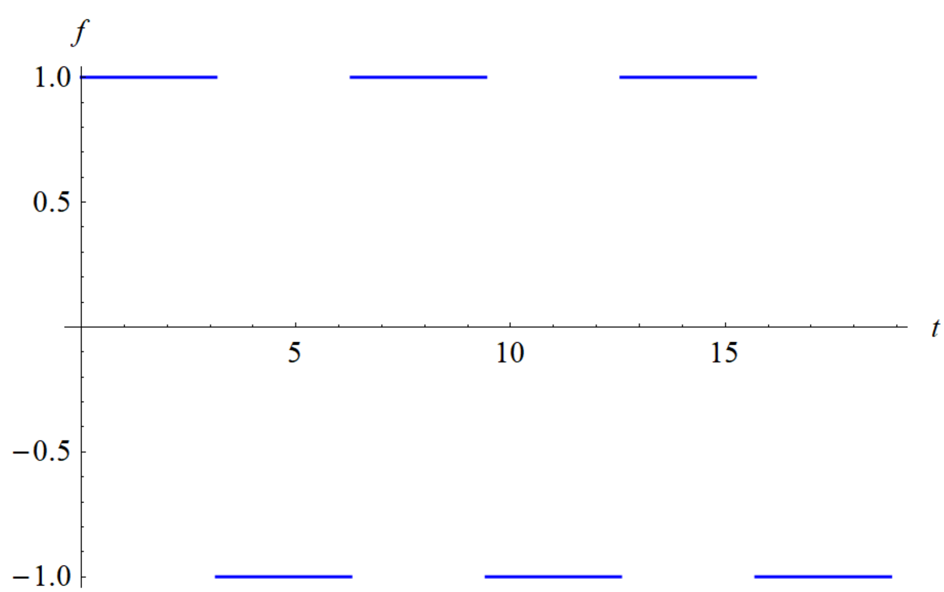
(c) Let $n = 15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does.

(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

Solution

On the interval $0 \le t \le 6\pi$, $u_{k\pi}(t)$ is nonzero if $k < 6$ and 0 if $k \ge 6$.

$$\begin{aligned} f(t) &= u_0(t) + 2\sum_{k=1}^{5}(-1)^k u_{k\pi}(t) + 2\sum_{k=6}^{n}(-1)^k u_{k\pi}(t) \\ &= u_0(t) + 2\sum_{k=1}^{5}(-1)^k u_{k\pi}(t) + 2\sum_{k=6}^{n}(-1)^k (0) \\ &= u_0(t) + 2\sum_{k=1}^{5}(-1)^k u_{k\pi}(t) \end{aligned}$$



Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Substitute the provided function for $f(t)$ and take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t)\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_0(t)\} + 2\sum_{k=1}^{n}(-1)^k \mathcal{L}\{u_{k\pi}(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \int_0^\infty e^{-st} u_0(t)\, dt + 2\sum_{k=1}^{n}(-1)^k \int_0^\infty e^{-st} u_{k\pi}(t)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + Y(s) = \int_0^\infty e^{-st}\, dt + 2\sum_{k=1}^{n}(-1)^k \int_{k\pi}^\infty e^{-st}\, dt$$

$$(s^2+1)Y(s) = \frac{1}{s} + 2\sum_{k=1}^{n}(-1)^k \left(\frac{e^{-k\pi s}}{s}\right)$$

Solve for $Y(s)$.

$$Y(s) = \frac{1}{s(s^2+1)} + 2\sum_{k=1}^{n}(-1)^k \left[\frac{1}{s(s^2+1)}\right] e^{-k\pi s}$$

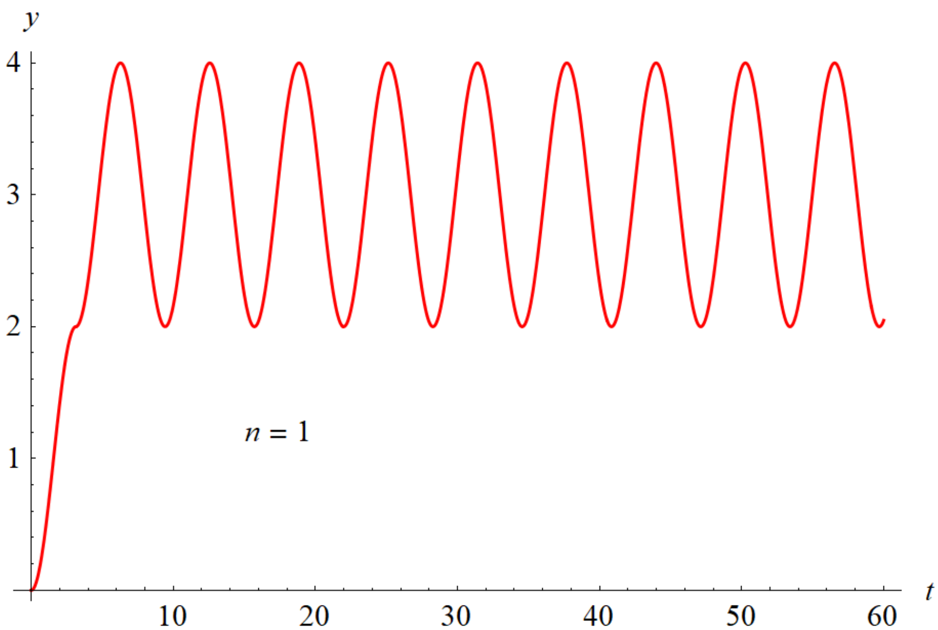
Now write it in terms of known transforms by using partial fraction decomposition.

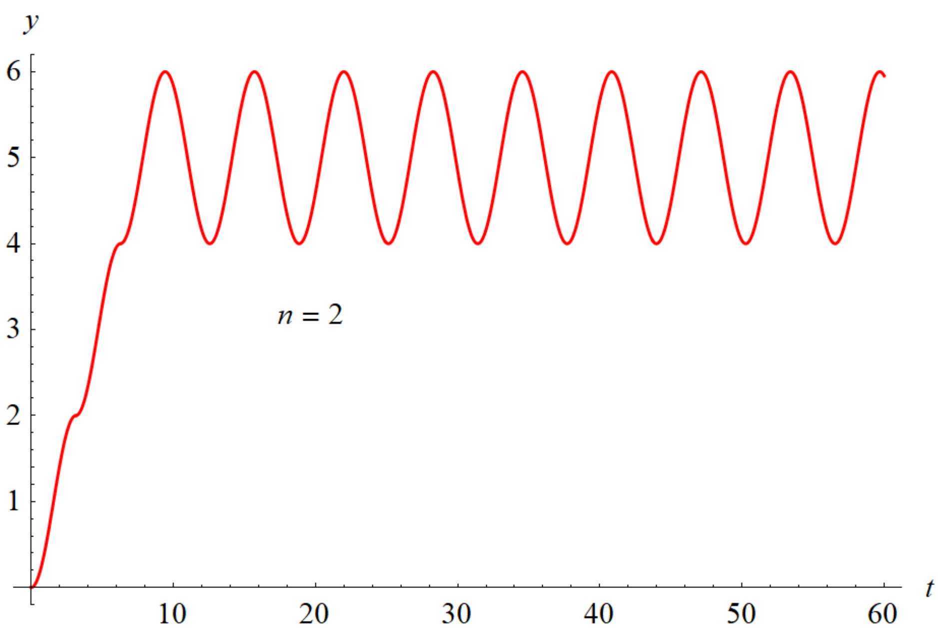
$$Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 2\sum_{k=1}^{n}(-1)^k \left(\frac{1}{s} - \frac{s}{s^2+1}\right) e^{-k\pi s}$$

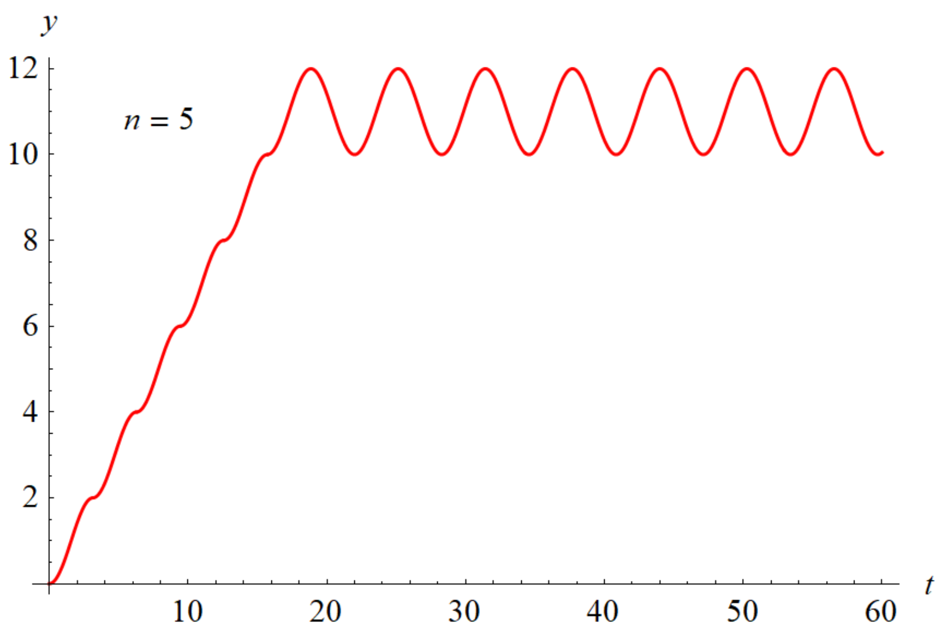
Take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

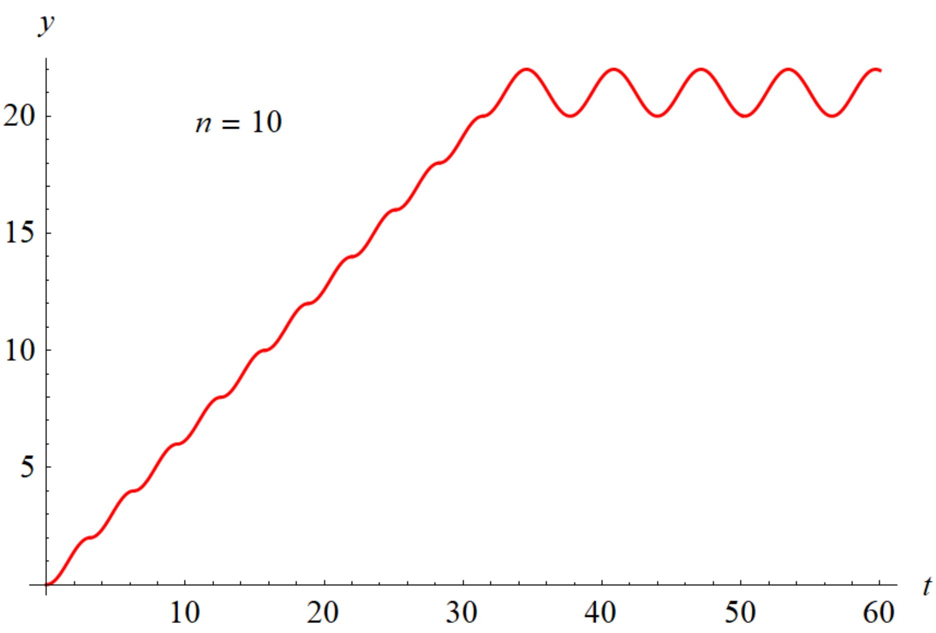
$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 2\sum_{k=1}^{n}(-1)^k\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-k\pi s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} + 2\sum_{k=1}^{n}(-1)^k\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-k\pi s}\right\} \\
&= 1 - \cos t + 2\sum_{k=1}^{n}(-1)^k[1-\cos(t-k\pi)]H(t-k\pi) \\
&= 1 - \cos t + 2\sum_{k=1}^{n}(-1)^k[1-\cos(t-k\pi)]u_{k\pi}(t)
\end{aligned}$$

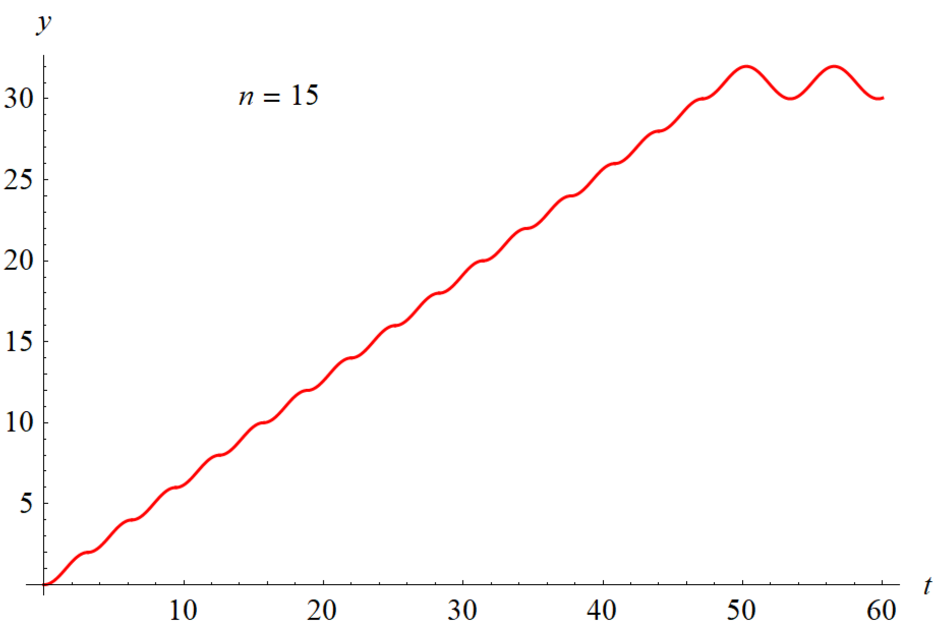
Graphs of $y(t)$ versus t are shown below for $n = 1$, $n = 2$, $n = 5$, $n = 10$, and $n = 15$.




y
6
5
4
3
2
1
10
20
30
40
50
60
t
n = 2


y
12
10
8
6
4
2
10
20
30
40
50
60
t
n = 5





The value of n determines the number of ridges as the graph increases. The graph oscillates with an amplitude of 1 and a period of 2π about the value $y = 2n + 1$. If $n \to \infty$, the graph grows at a linear rate forever.