Problem 22

Consider the initial value problem

$$y'' + 0.1y' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $g(t)$ is the same as in Problem 21.

(a) Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.

(b) Estimate the amplitude and frequency of the steady state part of the solution.

(c) Compare the results of part (b) with those from Problem 20 and from Section 3.8 for a sinusoidally forced oscillator.

Solution

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Substitute the function for $g(t)$ and take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + 0.1y' + y\} = \mathcal{L}\left\{u_0(t) + \sum_{k=1}^{n}(-1)^k u_{k\pi}(t)\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 0.1\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{u_0(t)\} + \sum_{k=1}^{n}(-1)^k \mathcal{L}\{u_{k\pi}(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 0.1[sY(s) - y(0)] + Y(s) = \int_0^\infty e^{-st} u_0(t)\, dt + \sum_{k=1}^{n}(-1)^k \int_0^\infty e^{-st} u_{k\pi}(t)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 0.1[sY(s)] + Y(s) = \int_0^\infty e^{-st}\, dt + \sum_{k=1}^{n}(-1)^k \int_{k\pi}^\infty e^{-st}\, dt$$

$$\left(s^2 + \frac{1}{10}s + 1\right) Y(s) = \frac{1}{s} + \sum_{k=1}^{n}(-1)^k \left(\frac{e^{-k\pi s}}{s}\right)$$

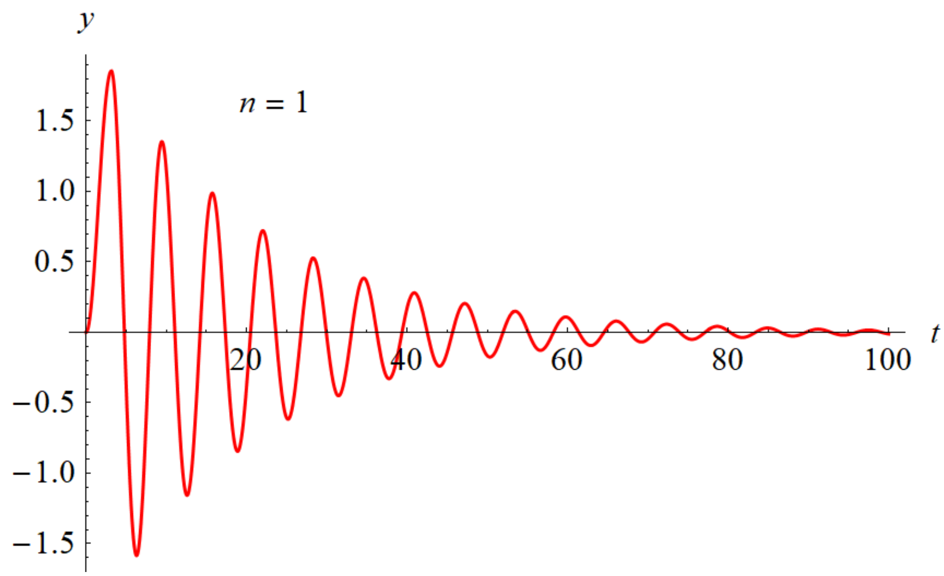
Solve for $Y(s)$ and then write it in terms of known transforms by using partial fraction decomposition and then completing the square.

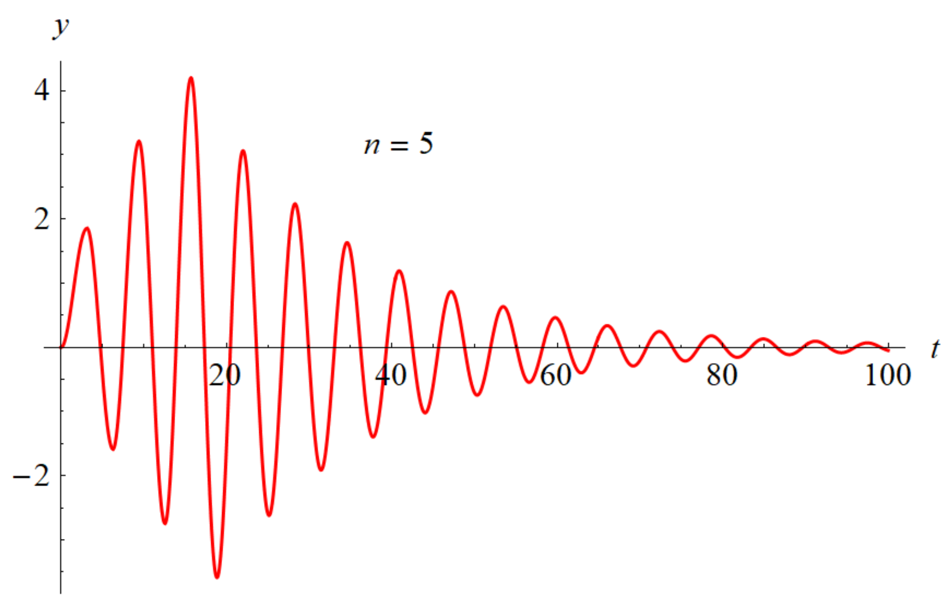
$$\begin{aligned}
Y(s) &= \frac{1}{s\left(s^2+\frac{1}{10}s+1\right)} + \sum_{k=1}^{n}(-1)^k\left[\frac{1}{s\left(s^2+\frac{1}{10}s+1\right)}\right]e^{-k\pi s} \\
&= \left(\frac{1}{s} + \frac{-s-\frac{1}{10}}{s^2+\frac{1}{10}s+1}\right) + \sum_{k=1}^{n}(-1)^k\left(\frac{1}{s} + \frac{-s-\frac{1}{10}}{s^2+\frac{1}{10}s+1}\right)e^{-k\pi s} \\
&= \left(\frac{1}{s} - \frac{s+\frac{1}{10}}{s^2+\frac{1}{10}s+\frac{1}{400}+1-\frac{1}{400}}\right) + \sum_{k=1}^{n}(-1)^k\left(\frac{1}{s} - \frac{s+\frac{1}{10}}{s^2+\frac{1}{10}s+\frac{1}{400}+1-\frac{1}{400}}\right)e^{-k\pi s} \\
&= \left[\frac{1}{s} - \frac{s+\frac{1}{10}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right] + \sum_{k=1}^{n}(-1)^k\left[\frac{1}{s} - \frac{s+\frac{1}{10}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right]e^{-k\pi s} \\
&= \left[\frac{1}{s} - \frac{\left(s+\frac{1}{20}\right)+\frac{1}{10}-\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right] + \sum_{k=1}^{n}(-1)^k\left[\frac{1}{s} - \frac{\left(s+\frac{1}{20}\right)+\frac{1}{10}-\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right]e^{-k\pi s} \\
&= \left[\frac{1}{s} - \frac{\left(s+\frac{1}{20}\right)+\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right] + \sum_{k=1}^{n}(-1)^k\left[\frac{1}{s} - \frac{\left(s+\frac{1}{20}\right)+\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right]e^{-k\pi s} \\
&= \left[\frac{1}{s} - \frac{s+\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}} - \frac{\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right] + \sum_{k=1}^{n}(-1)^k\left[\frac{1}{s} - \frac{s+\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}} - \frac{\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right]e^{-k\pi s} \\
&= \left[\frac{1}{s} - \frac{s+\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}} - \frac{1}{\sqrt{399}}\frac{\frac{\sqrt{399}}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right] \\
&\qquad + \sum_{k=1}^{n}(-1)^k\left[\frac{1}{s} - \frac{s+\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}} - \frac{1}{\sqrt{399}}\frac{\frac{\sqrt{399}}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right]e^{-k\pi s}
\end{aligned}$$

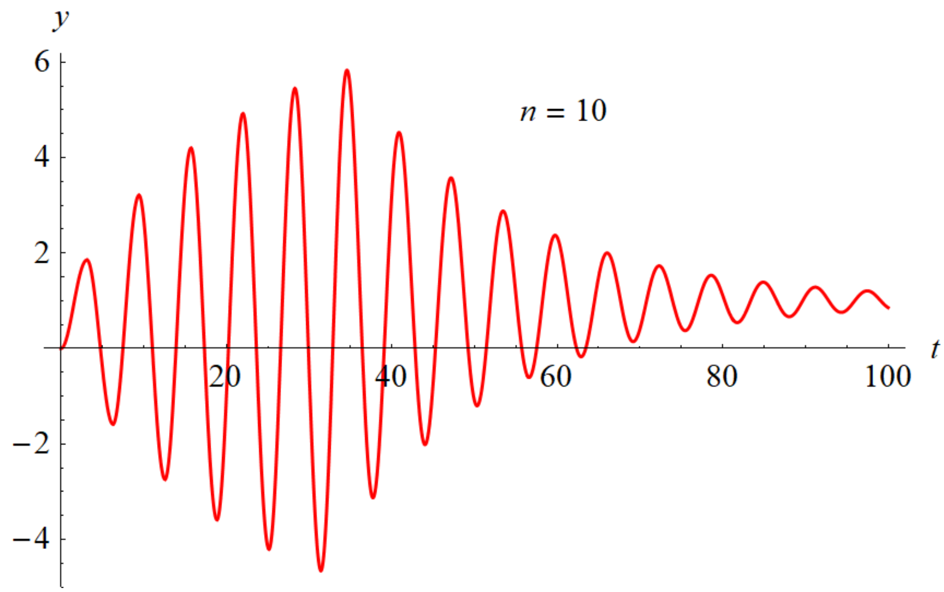
Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

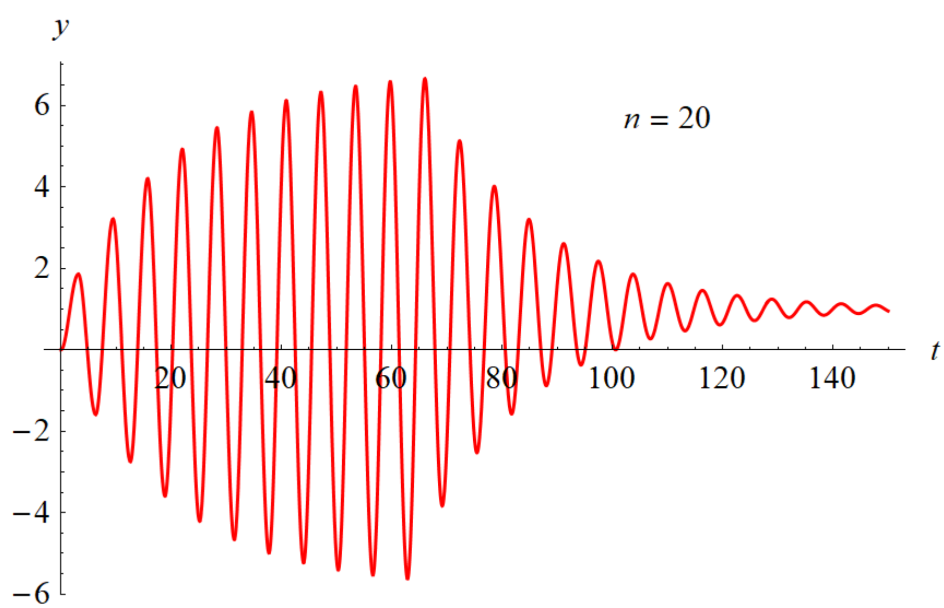
$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\left[\frac{1}{s} - \frac{s+\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}} - \frac{1}{\sqrt{399}}\frac{\frac{\sqrt{399}}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right]\right. \\
&\qquad \left. + \sum_{k=1}^{n}(-1)^k\left[\frac{1}{s} - \frac{s+\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}} - \frac{1}{\sqrt{399}}\frac{\frac{\sqrt{399}}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right]e^{-k\pi s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}} - \frac{1}{\sqrt{399}}\frac{\frac{\sqrt{399}}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right\} \\
&\qquad + \sum_{k=1}^{n}(-1)^k\mathcal{L}^{-1}\left\{\left[\frac{1}{s} - \frac{s+\frac{1}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}} - \frac{1}{\sqrt{399}}\frac{\frac{\sqrt{399}}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}\right]e^{-k\pi s}\right\} \\
&= 1 - e^{-t/20}\cos\frac{\sqrt{399}}{20}t - \frac{1}{\sqrt{399}}e^{-t/20}\sin\frac{\sqrt{399}}{20}t \\
&\quad + \sum_{k=1}^{n}(-1)^k\left\{1 - e^{-(t-k\pi)/20}\cos\left[\frac{\sqrt{399}}{20}(t-k\pi)\right] - \frac{1}{\sqrt{399}}e^{-(t-k\pi)/20}\sin\left[\frac{\sqrt{399}}{20}(t-k\pi)\right]\right\}H(t-k\pi)
\end{aligned}$$

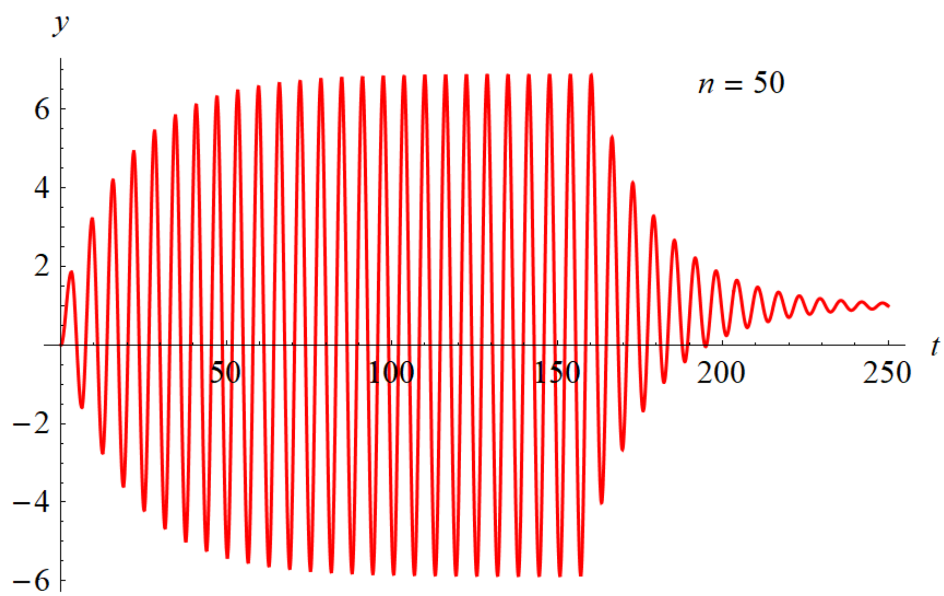
Below are several graphs of $y(t)$ versus t for various values of n.






y
6
4
2
−2
−4
n = 10
20
40
60
80
100
t

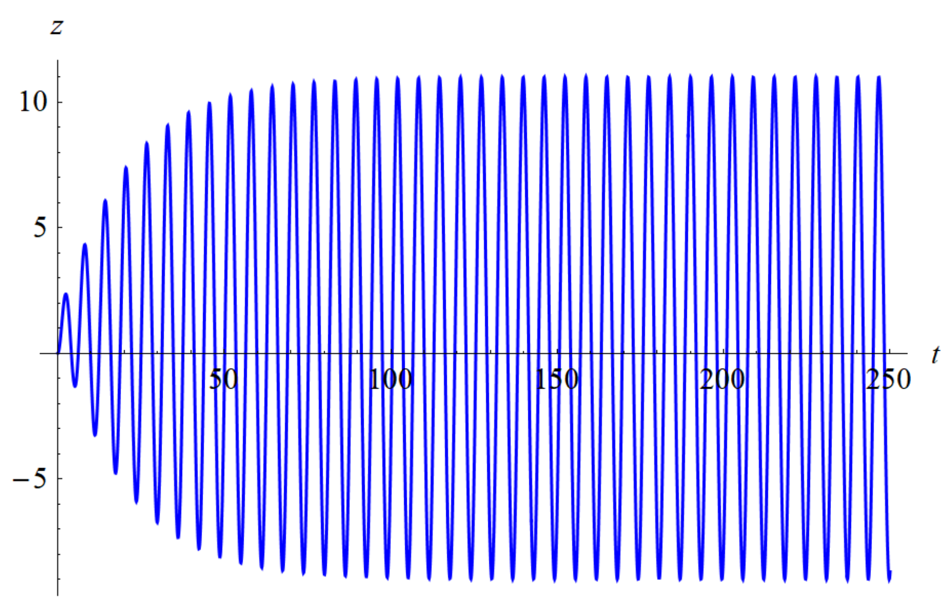

y
6
4
2
−2
−4
−6
n = 20
20
40
60
80
100
120
140
t



The amplitude of the steady part of the solution is about 6.8, and the period is $T = 2\pi$, which means the angular frequency is $\omega = 2\pi/T = 1$. This amplitude is significantly less than the one found in Problem 20 for the forcing function $f(t)$. The corresponding problem with a sinusoidal forcing term was considered in Section 3.8. Note that $1 + \cos t$ most closely resembles $g(t)$.

$$z'' + 0.1z' + z = 1 + \cos t, \qquad z(0) = 0, \quad z'(0) = 0$$

The solution to this ODE is plotted below.



The amplitude of the steady part of the solution here is about 10.9, and the period is $T = 2\pi$, which means the angular frequency is $\omega = 2\pi/T = 1$. This difference in amplitude is due to the fact that the cosine forcing varies in strength with time, whereas $g(t)$ delivers pulses at maximum strength once in a while.