Problem 7

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + y = u_{3\pi}(t); \qquad y(0) = 1, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{u_{3\pi}(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_{3\pi}(t)\}$$
$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \int_0^\infty e^{-st}[u_{3\pi}(t)]\, dt$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = 0$.

$$[s^2Y(s) - s] + Y(s) = \int_{3\pi}^\infty e^{-st}\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$\begin{aligned}(s^2+1)Y(s) - s &= \left(-\frac{1}{s}e^{-st}\right)\Bigg|_{3\pi}^\infty \\ &= \frac{1}{s}e^{-3\pi s}\end{aligned}$$

$$(s^2+1)Y(s) = \frac{1}{s}e^{-3\pi s} + s$$

$$\begin{aligned}Y(s) &= \frac{1}{s(s^2+1)}e^{-3\pi s} + \frac{s}{s^2+1} \\ &= \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-3\pi s} + \frac{s}{s^2+1}\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-3\pi s} + \frac{s}{s^2+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-3\pi s}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \\
&= [1 - \cos(t-3\pi)]H(t-3\pi) + \cos t \\
&= [1 - \cos(t-\pi)]H(t-3\pi) + \cos t \\
&= (1+\cos t)H(t-3\pi) + \cos t \\
&= (1+\cos t)u_{3\pi}(t) + \cos t
\end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $f(t)$ versus t.