Problem 1

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$y'' + 2y' + 2y = \delta(t - \pi); \qquad y(0) = 1, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\delta(t - \pi)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t - \pi)\}$$
$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2[Y(s)] = \int_0^\infty e^{-st}[\delta(t - \pi)]\, dt$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = 0$.

$$[s^2Y(s) - s] + 2[sY(s) - 1] + 2[Y(s)] = e^{-s(\pi)}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 2s + 2)Y(s) - s - 2 = e^{-\pi s}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$Y(s) = \frac{2}{s^2 + 2s + 2} + \frac{s}{s^2 + 2s + 2} + \frac{1}{s^2 + 2s + 2}e^{-\pi s}$$

Complete the square in the denominators.

$$\begin{aligned} Y(s) &= \frac{2}{s^2 + 2s + 1 + 2 - 1} + \frac{s}{s^2 + 2s + 1 + 2 - 1} + \frac{1}{s^2 + 2s + 1 + 2 - 1}e^{-\pi s} \\ &= \frac{2}{(s+1)^2 + 1} + \frac{s}{(s+1)^2 + 1} + \frac{1}{(s+1)^2 + 1}e^{-\pi s} \end{aligned}$$

Make it so that $s+1$ appears in the numerator.

$$\begin{aligned} Y(s) &= \frac{2}{(s+1)^2+1} + \frac{(s+1)-1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s} \\ &= \frac{2}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s} \\ &= \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s} \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}e^{-\pi s}\right\} \\ &= e^{-t}\sin t + e^{-t}\cos t + e^{-(t-\pi)}\sin(t-\pi)H(t-\pi) \\ &= e^{-t}(\sin t + \cos t) + e^{\pi-t}(-\sin t)H(t-\pi) \\ &= e^{-t}(\sin t + \cos t) - e^{\pi-t}(\sin t)u_\pi(t) \end{aligned}$$

Below is a plot of $y(t)$ versus t.