Problem 4

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$y'' - y = -20\delta(t-3); \qquad y(0) = 1, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' - y\} = \mathcal{L}\{-20\delta(t-3)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = -20\mathcal{L}\{\delta(t-3)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] - [Y(s)] = -20\int_0^\infty e^{-st}[\delta(t-3)]\, dt$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = 0$.

$$[s^2Y(s) - s] - [Y(s)] = -20e^{-s(3)}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 - 1)Y(s) - s = -20e^{-3s}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$Y(s) = \frac{s}{s^2-1} - \frac{20}{s^2-1}e^{-3s}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{s}{s^2-1} - \frac{20}{s^2-1}e^{-3s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} - 20\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}e^{-3s}\right\} \\ &= \cosh t - 20\sinh(t-3)H(t-3) \\ &= \cosh t - 20\sinh(t-3)u_3(t) \end{aligned}$$

Below is a plot of $y(t)$ versus t.