Problem 6

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$y'' + 4y = \delta(t - 4\pi); \qquad y(0) = 1/2, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{\delta(t - 4\pi)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t - 4\pi)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 4[Y(s)] = \int_0^\infty e^{-st}[\delta(t - 4\pi)]\, dt$$

Plug in the initial conditions, $y(0) = 1/2$ and $y'(0) = 0$.

$$[s^2Y(s) - s/2] + 4[Y(s)] = e^{-s(4\pi)}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 4)Y(s) - \frac{1}{2}s = e^{-4\pi s}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{1}{2}\frac{s}{s^2 + 4} + \frac{1}{s^2 + 4}e^{-4\pi s} \\ &= \frac{1}{2}\frac{s}{s^2 + 4} + \frac{1}{2}\frac{2}{s^2 + 4}e^{-4\pi s} \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}e^{-4\pi s}\right\} \\
&= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}e^{-4\pi s}\right\} \\
&= \frac{1}{2}\cos 2t + \frac{1}{2}[\sin 2(t-4\pi)]H(t-4\pi) \\
&= \frac{1}{2}\cos 2t + \frac{1}{2}(\sin 2t)H(t-4\pi) \\
&= \frac{1}{2}[\cos 2t + H(t-4\pi)\sin 2t] \\
&= \frac{1}{2}[\cos 2t + u_{4\pi}(t)\sin 2t]
\end{aligned}$$

Below is a plot of $y(t)$ versus t.