Problem 7

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$y'' + y = \delta(t - 2\pi)\cos t; \qquad y(0) = 0, \quad y'(0) = 1$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\delta(t - 2\pi)\cos t\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t - 2\pi)\cos t\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + [Y(s)] = \int_0^\infty e^{-st}[\delta(t - 2\pi)\cos t]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$[s^2Y(s) - 1] + [Y(s)] = e^{-s(2\pi)}\cos(2\pi)$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 1)Y(s) - 1 = e^{-2\pi s}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$Y(s) = \frac{1}{s^2 + 1} + \frac{1}{s^2 + 1}e^{-2\pi s}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} + \frac{1}{s^2+1}e^{-2\pi s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-2\pi s}\right\} \\ &= \sin t + \sin(t-2\pi)H(t-2\pi) \\ &= \sin t + (\sin t)H(t-2\pi) \\ &= [1+H(t-2\pi)]\sin t \\ &= [1+u_{2\pi}(t)]\sin t \end{aligned}$$

Below is a plot of $y(t)$ versus t.