Problem 10

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$2y'' + y' + 4y = \delta(t - \pi/6)\sin t; \qquad y(0) = 0, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{2y'' + y' + 4y\} = \mathcal{L}\{\delta(t - \pi/6)\sin t\}$$

Use the fact that the transform is a linear operator.

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t - \pi/6)\sin t\}$$
$$2[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + 4[Y(s)] = \int_0^\infty e^{-st}[\delta(t - \pi/6)\sin t]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$2[s^2Y(s)] + [sY(s)] + 4[Y(s)] = e^{-s(\pi/6)}\sin(\pi/6)$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(2s^2 + s + 4)Y(s) = \frac{1}{2}e^{-\pi s/6}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned}
Y(s) &= \frac{1}{2(2s^2 + s + 4)}e^{-\pi s/6} \\
&= \frac{1}{4\left(s^2 + \frac{1}{2}s + 2\right)}e^{-\pi s/6} \\
&= \frac{1}{4\left(s^2 + \frac{1}{2}s + \frac{1}{16} + 2 - \frac{1}{16}\right)}e^{-\pi s/6} \\
&= \frac{1}{4\left[\left(s + \frac{1}{4}\right)^2 + \frac{31}{16}\right]}e^{-\pi s/6} \\
&= \frac{1}{\sqrt{31}} \frac{\frac{\sqrt{31}}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{31}{16}}e^{-\pi s/6}
\end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{31}}\frac{\frac{\sqrt{31}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{31}{16}}e^{-\pi s/6}\right\} \\
&= \frac{1}{\sqrt{31}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{31}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{31}{16}}e^{-\pi s/6}\right\} \\
&= \frac{1}{\sqrt{31}}e^{-(t-\pi/6)/4}\sin\left[\frac{\sqrt{31}}{4}(t-\pi/6)\right]H(t-\pi/6) \\
&= \frac{1}{\sqrt{31}}e^{(\pi-6t)/24}\sin\left[\frac{\sqrt{31}}{24}(6t-\pi)\right]u_{\pi/6}(t)
\end{aligned}$$

Below is a plot of $y(t)$ versus t.