Problem 14

Consider the initial value problem

$$y'' + \gamma y' + y = \delta(t-1), \qquad y(0) = 0, \quad y'(0) = 0,$$

where γ is the damping coefficient (or resistance).

(a) Let $\gamma = \frac{1}{2}$. Find the solution of the initial value problem and plot its graph.

(b) Find the time t_1 at which the solution attains its maximum value. Also find the maximum value y_1 of the solution.

(c) Let $\gamma = \frac{1}{4}$ and repeat parts (a) and (b).

(d) Determine how t_1 and y_1 vary as γ decreases. What are the values of t_1 and y_1 when $\gamma = 0$?

Solution

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + \gamma y' + y\} = \mathcal{L}\{\delta(t-1)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \gamma\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-1)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + \gamma[sY(s) - y(0)] + Y(s) = \int_0^\infty e^{-st}\delta(t-1)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + \gamma[sY(s)] + Y(s) = e^{-s(1)}$$

$$(s^2 + \gamma s + 1)Y(s) = e^{-s}$$

Solve for $Y(s)$.

$$Y(s) = \frac{1}{s^2 + \gamma s + 1} e^{-s}$$

Suppose that $0 < \gamma < 2$.

$$\begin{aligned} Y(s) &= \frac{1}{s^2 + \gamma s + \frac{\gamma^2}{4} + 1 - \frac{\gamma^2}{4}} e^{-s} \\ &= \frac{1}{\left(s + \frac{\gamma}{2}\right)^2 + \frac{4-\gamma^2}{4}} e^{-s} \\ &= \frac{2}{\sqrt{4-\gamma^2}} \frac{\frac{\sqrt{4-\gamma^2}}{2}}{\left(s + \frac{\gamma}{2}\right)^2 + \frac{4-\gamma^2}{4}} e^{-s} \end{aligned}$$

Take the inverse Laplace transform to get $y(t)$.

$$y(t) = \frac{2}{\sqrt{4-\gamma^2}} e^{-\gamma(t-1)/2} \sin\left[\frac{\sqrt{4-\gamma^2}}{2}(t-1)\right] H(t-1)$$

For values of $t > 1$, the Heaviside function is 1.

$$y(t) = \frac{2}{\sqrt{4-\gamma^2}} e^{-\gamma(t-1)/2} \sin\left[\frac{\sqrt{4-\gamma^2}}{2}(t-1)\right], \quad t > 1$$

Take the derivative and set it equal to zero to find the value of t for which $y(t)$ is maximum.

$$y'(t) = -\frac{\gamma}{\sqrt{4-\gamma^2}} e^{-\gamma(t-1)/2} \sin\left[\frac{\sqrt{4-\gamma^2}}{2}(t-1)\right] + ke^{-\gamma(t-1)/2} \cos\left[\frac{\sqrt{4-\gamma^2}}{2}(t-1)\right] = 0$$

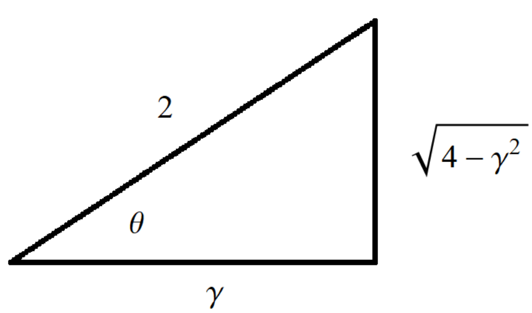
$$\tan\left[\frac{\sqrt{4-\gamma^2}}{2}(t-1)\right] = \frac{\sqrt{4-\gamma^2}}{\gamma}$$

$$t_1 = 1 + \frac{2}{\sqrt{4-\gamma^2}} \tan^{-1}\left(\frac{\sqrt{4-\gamma^2}}{\gamma}\right)$$

Now plug this value of t into $y(t)$ to find the maximum value of y.

$$y(t_1) = y_1 = \frac{2k}{\sqrt{4-\gamma^2}} \exp\left[-\frac{\gamma}{\sqrt{4-\gamma^2}} \tan^{-1}\left(\frac{\sqrt{4-\gamma^2}}{\gamma}\right)\right] \sin\tan^{-1}\left(\frac{\sqrt{4-\gamma^2}}{\gamma}\right)$$

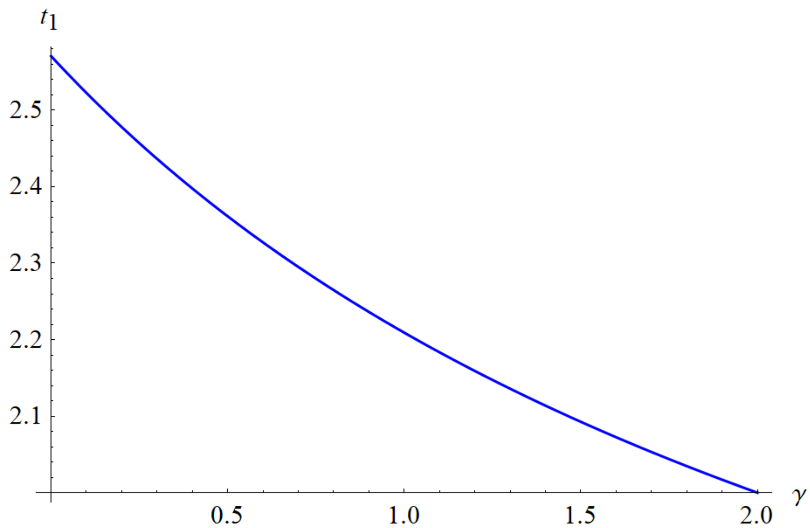
In order to determine the sine of the inverse tangent, draw the implied right triangle.

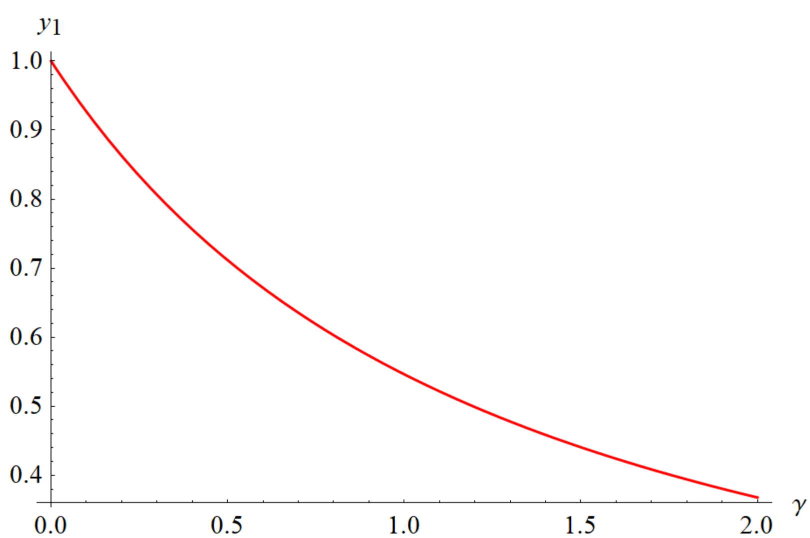


We see that the sine is $\sqrt{4-\gamma^2}/2$.

$$\begin{aligned} y_1 &= \frac{2}{\sqrt{4-\gamma^2}} \exp\left[-\frac{\gamma}{\sqrt{4-\gamma^2}} \tan^{-1}\left(\frac{\sqrt{4-\gamma^2}}{\gamma}\right)\right] \frac{\sqrt{4-\gamma^2}}{2} \\ &= \exp\left[-\frac{\gamma}{\sqrt{4-\gamma^2}} \tan^{-1}\left(\frac{\sqrt{4-\gamma^2}}{\gamma}\right)\right] \end{aligned}$$

Below are plots of t_1 versus γ and y_1 versus γ.



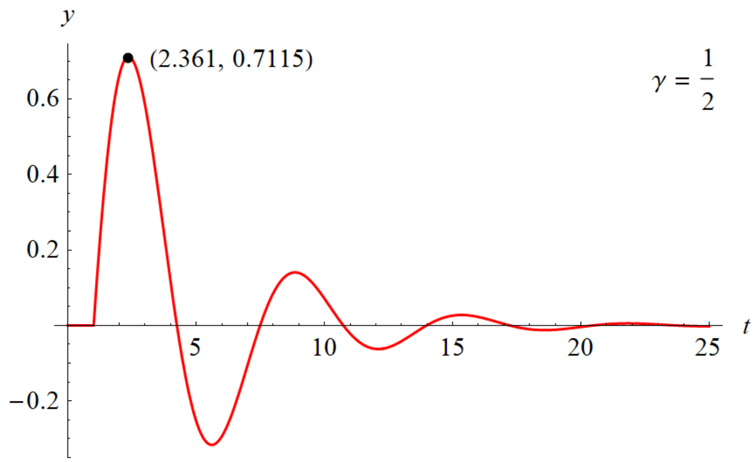


Note that

$$\begin{aligned} \lim_{\gamma\to 0} t_1(\gamma) &= 1 + \frac{\pi}{2} \approx 2.57 \\ \lim_{\gamma\to 0} y_1(\gamma) &= 1. \end{aligned}$$

In particular, if $\gamma = 1/2$, then

$$
\begin{aligned}
y(t) &= \frac{4}{\sqrt{15}} e^{-(t-1)/4} \sin\left[\frac{\sqrt{15}}{4}(t-1)\right] H(t-1) \\
t_1 &= 1 + \frac{4}{\sqrt{15}} \tan^{-1}\sqrt{15} \approx 2.361 \\
y_1 &= \exp\left(-\frac{1}{\sqrt{15}} \tan^{-1}\sqrt{15}\right) \approx 0.7115,
\end{aligned}
$$



and if $\gamma = 1/4$, then

$$
\begin{aligned}
y(t) &= \frac{8}{\sqrt{63}} e^{-(t-1)/8} \sin\left[\frac{\sqrt{63}}{8}(t-1)\right] H(t-1) \\
t_1 &= 1 + \frac{8}{\sqrt{63}} \tan^{-1}\sqrt{63} \approx 2.457 \\
y_1 &= \exp\left(-\frac{1}{\sqrt{63}} \tan^{-1}\sqrt{63}\right) \approx 0.8335.
\end{aligned}
$$