Problem 24

Proceed as in Problem 23 for the oscillator satisfying

$$y'' + 0.1y' + y = \sum_{k=1}^{15} \delta[t - (2k-1)\pi], \qquad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 21.

Solution

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + 0.1y' + y\} = \mathcal{L}\left\{\sum_{k=1}^{15} \delta[t - (2k-1)\pi]\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 0.1\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \sum_{k=1}^{15} \mathcal{L}\{\delta[t - (2k-1)\pi]\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 0.1[sY(s) - y(0)] + Y(s) = \sum_{k=1}^{15} \int_0^\infty e^{-st}\delta[t - (2k-1)\pi]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 0.1[sY(s)] + Y(s) = \sum_{k=1}^{15} e^{-s(2k-1)\pi}$$

$$\left(s^2 + \frac{1}{10}s + 1\right) Y(s) = \sum_{k=1}^{15} e^{-(2k-1)\pi s}$$

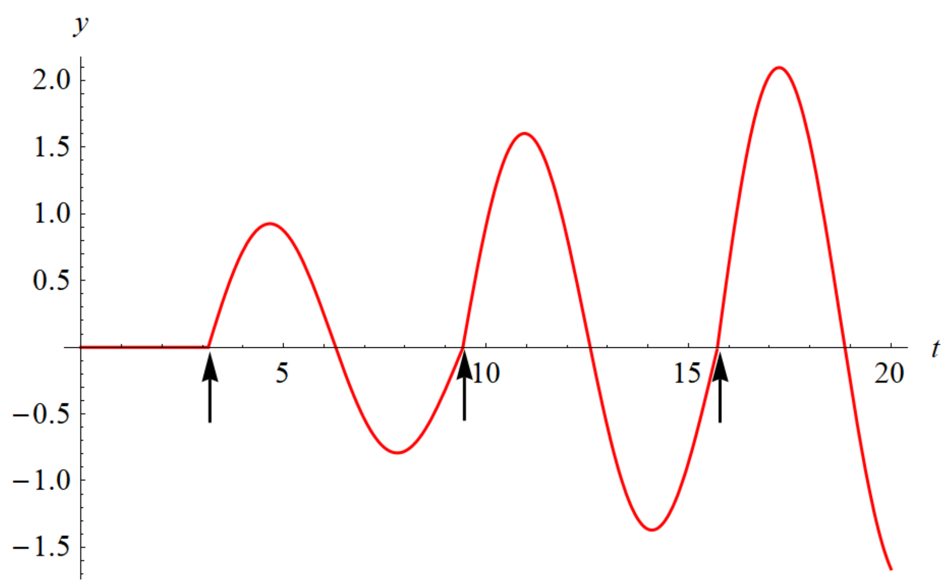
Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned}
Y(s) &= \sum_{k=1}^{15} \frac{1}{s^2 + \frac{1}{10}s + 1} e^{-(2k-1)\pi s} \\
&= \sum_{k=1}^{15} \frac{1}{s^2 + \frac{1}{10}s + \frac{1}{400} + 1 - \frac{1}{400}} e^{-(2k-1)\pi s} \\
&= \sum_{k=1}^{15} \frac{1}{\left(s + \frac{1}{20}\right)^2 + \frac{399}{400}} e^{-(2k-1)\pi s} \\
&= \frac{20}{\sqrt{399}} \sum_{k=1}^{15} \frac{\frac{\sqrt{399}}{20}}{\left(s + \frac{1}{20}\right)^2 + \frac{399}{400}} e^{-(2k-1)\pi s}
\end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

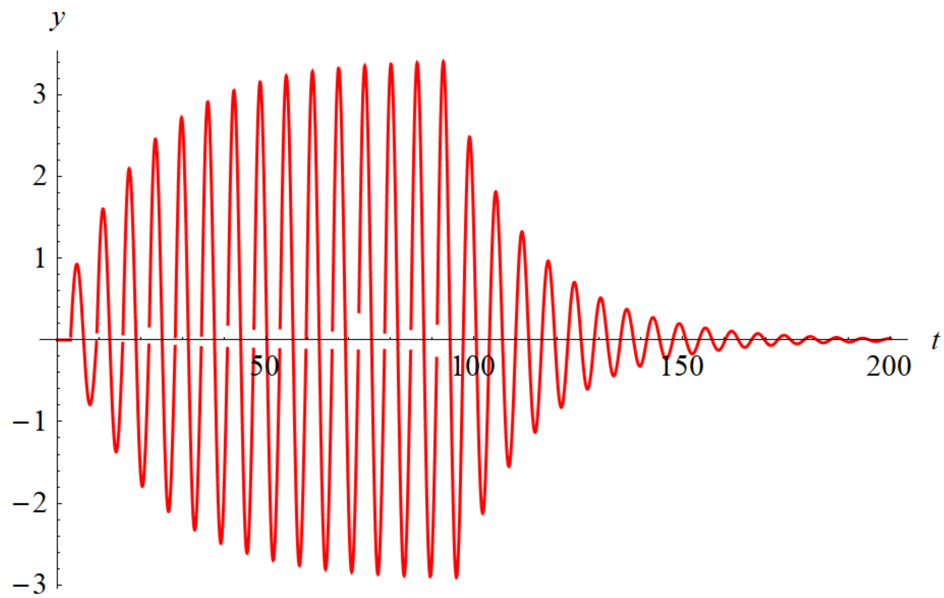
$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{ \frac{20}{\sqrt{399}} \sum_{k=1}^{15} \frac{\frac{\sqrt{399}}{20}}{\left(s + \frac{1}{20}\right)^2 + \frac{399}{400}} e^{-(2k-1)\pi s} \right\} \\
&= \frac{20}{\sqrt{399}} \sum_{k=1}^{15} \mathcal{L}^{-1}\left\{ \frac{\frac{\sqrt{399}}{20}}{\left(s + \frac{1}{20}\right)^2 + \frac{399}{400}} e^{-(2k-1)\pi s} \right\} \\
&= \frac{20}{\sqrt{399}} \sum_{k=1}^{15} e^{-[t-(2k-1)\pi]/20} \sin\left\{ \frac{\sqrt{399}}{20}[t - (2k-1)\pi] \right\} H[t - (2k-1)\pi] \\
&= \frac{20}{\sqrt{399}} \sum_{k=1}^{15} e^{-[t-(2k-1)\pi]/20} \sin\left\{ \frac{\sqrt{399}}{20}[t - (2k-1)\pi] \right\} u_{(2k-1)\pi}(t)
\end{aligned}$$

Below is a plot of $y(t)$ versus t up until $t = 20$.



The arrows indicate the time, magnitude, and direction that the delta functions strike.

Below is a plot of $y(t)$ versus t up until $t = 200$.



After the delta function impulses stop, the amplitude of oscillation eventually decays to zero because of the y' term in the ODE.