Problem 11

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$y'' + 2y' + 2y = \cos t + \delta(t - \pi/2); \qquad y(0) = 0, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\cos t + \delta(t - \pi/2)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\cos t\} + \mathcal{L}\{\delta(t - \pi/2)\sin t\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2[Y(s)] = \frac{s}{s^2+1} + \int_0^\infty e^{-st}[\delta(t - \pi/2)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 2[sY(s)] + 2[Y(s)] = \frac{s}{s^2+1} + e^{-s(\pi/2)}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 2s + 2)Y(s) = \frac{s}{s^2+1} + e^{-\pi s/2}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{1}{s^2+2s+2}e^{-\pi s/2}$$

Use partial fraction decomposition.

$$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

Multiply both sides by $(s^2+1)(s^2+2s+2)$.

$$s = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1)$$

Plug in four random values for s to get a system of four equations for A, B, C, and D.

$$\begin{aligned}
s=0: &\quad 0 = 2B + D \\
s=1: &\quad 1 = 5A + 5B + 2C + 2D \\
s=2: &\quad 2 = 20A + 10B + 10C + 5D \\
s=3: &\quad 3 = 51A + 17B + 30C + 10D
\end{aligned}$$

Solving this system yields $A = 1/5$, $B = 2/5$, $C = -1/5$, and $D = -4/5$.

$$Y(s) = \left(\frac{\frac{1}{5}s + \frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}s - \frac{4}{5}}{s^2+2s+2}\right) + \frac{1}{s^2+2s+2}e^{-\pi s/2}$$

Complete the square in the denominators.

$$\begin{aligned}
Y(s) &= \frac{\frac{1}{5}s + \frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}s - \frac{4}{5}}{s^2+2s+1+2-1} + \frac{1}{s^2+2s+1+2-1}e^{-\pi s/2} \\
&= \frac{\frac{1}{5}s + \frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}s - \frac{4}{5}}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s/2}
\end{aligned}$$

Make it so that $s+1$ appears in the numerator.

$$\begin{aligned}
Y(s) &= \frac{\frac{1}{5}s + \frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}(s+1) + \frac{1}{5} - \frac{4}{5}}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s/2} \\
&= \frac{\frac{1}{5}s + \frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}(s+1) - \frac{3}{5}}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s/2} \\
&= \frac{1}{5}\frac{s}{s^2+1} + \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s+1}{(s+1)^2+1} - \frac{3}{5}\frac{1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s/2}
\end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{s}{s^2+1} + \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s+1}{(s+1)^2+1} - \frac{3}{5}\frac{1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s/2}\right\} \\
&= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} \\
&\qquad - \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}e^{-\pi s/2}\right\} \\
&= \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t + [e^{-(t-\pi/2)}\sin(t-\pi/2)]H(t-\pi/2) \\
&= \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t + e^{\pi/2-t}(-\cos t)H(t-\pi/2) \\
&= \frac{1}{5}[\cos t + 2\sin t - e^{-t}(\cos t + 3\sin t)] - u_{\pi/2}(t)e^{\pi/2-t}\cos t
\end{aligned}$$

Below is a plot of $y(t)$ versus t.