

Problem 16

Consider the initial value problem

$$y'' + y = f_k(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where $f_k(t) = [u_{4-k}(t) - u_{4+k}(t)]/2k$ with $0 < k \leq 1$.

- (a) Find the solution $y = \phi(t, k)$ of the initial value problem.
- (b) Calculate $\lim_{k \rightarrow 0^+} \phi(t, k)$ from the solution found in part (a).
- (c) Observe that $\lim_{k \rightarrow 0^+} f_k(t) = \delta(t - 4)$. Find the solution $\phi_0(t)$ of the given initial value problem with $f_k(t)$ replaced by $\delta(t - 4)$. Is it true that $\phi_0(t) = \lim_{k \rightarrow 0^+} \phi(t, k)$?
- (d) Plot $\phi(t, 1/2)$, $\phi(t, 1/4)$, and $\phi_0(t)$ on the same axes. Describe the relation between $\phi(t, k)$ and $\phi_0(t)$.