Problem 18

Problems 17 through 22 deal with the effect of a sequence of impulses on an undamped oscillator. Suppose that

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

For each of the following choices for $f(t)$:

(a) Try to predict the nature of the solution without solving the problem.

(b) Test your prediction by finding the solution and drawing its graph.

(c) Determine what happens after the sequence of impulses ends.

$$f(t) = \sum_{k=1}^{20} (-1)^{k+1} \delta(t - k\pi)$$

Solution

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Substitute the function for $f(t)$ and take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{\sum_{k=1}^{20} (-1)^{k+1} \delta(t - k\pi)\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \sum_{k=1}^{20} (-1)^{k+1} \mathcal{L}\{\delta(t - k\pi)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} \int_0^\infty e^{-st} \delta(t - k\pi)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-s(k\pi)}$$

$$(s^2 + 1)Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-k\pi s}$$

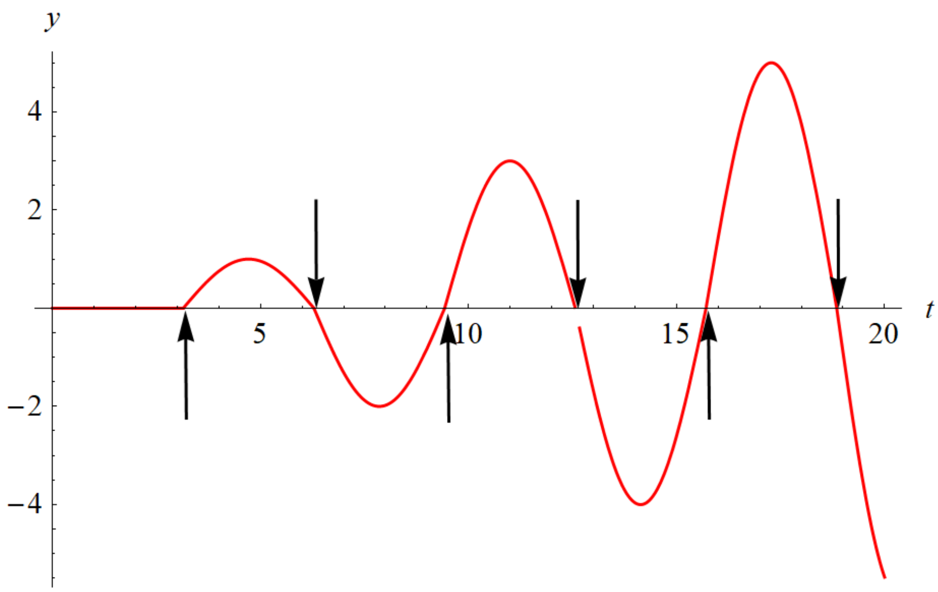
Solve for $Y(s)$.

$$Y(s) = \sum_{k=1}^{20} (-1)^{k+1} \frac{1}{s^2+1} e^{-k\pi s}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

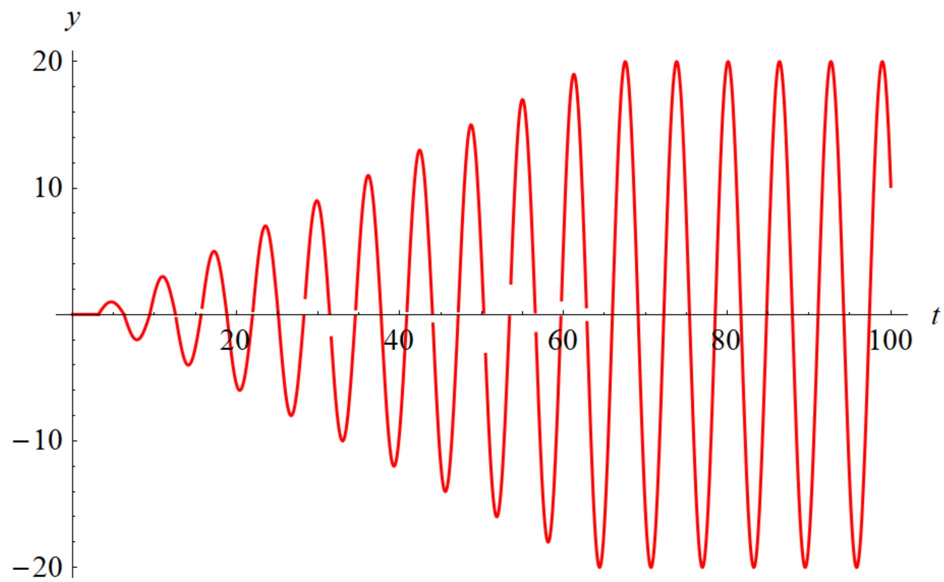
$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\sum_{k=1}^{20} (-1)^{k+1} \frac{1}{s^2+1} e^{-k\pi s}\right\} \\
&= \sum_{k=1}^{20} (-1)^{k+1} \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} e^{-k\pi s}\right\} \\
&= \sum_{k=1}^{20} (-1)^{k+1} \sin(t-k\pi) H(t-k\pi) \\
&= \sum_{k=1}^{20} (-1)^{k+1} \sin(t-k\pi) u_{k\pi}(t)
\end{aligned}$$

Below is a plot of $y(t)$ versus t up until $t = 20$.



The arrows indicate the time, magnitude, and direction that the delta functions strike.

Below is a plot of $y(t)$ versus t up until $t = 100$.



Once the delta functions stop, the solution oscillates with a constant amplitude because there's no term with y' in the ODE.