Problem 21

Problems 17 through 22 deal with the effect of a sequence of impulses on an undamped oscillator. Suppose that

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

For each of the following choices for $f(t)$:

(a) Try to predict the nature of the solution without solving the problem.

(b) Test your prediction by finding the solution and drawing its graph.

(c) Determine what happens after the sequence of impulses ends.

$$f(t) = \sum_{k=1}^{15} \delta[t - (2k-1)\pi]$$

Solution

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Substitute the function for $f(t)$ and take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{\sum_{k=1}^{15} \delta[t - (2k-1)\pi]\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \sum_{k=1}^{15} \mathcal{L}\{\delta[t - (2k-1)\pi]\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \sum_{k=1}^{15} \int_0^\infty e^{-st}\delta[t - (2k-1)\pi]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + Y(s) = \sum_{k=1}^{15} e^{-s(2k-1)\pi}$$

$$(s^2+1)Y(s) = \sum_{k=1}^{15} e^{-(2k-1)\pi s}$$

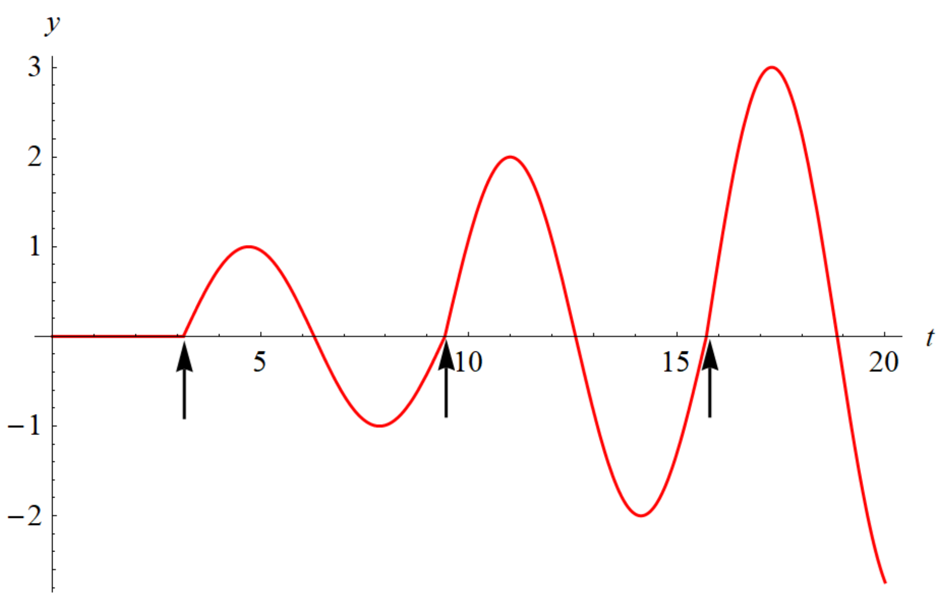
Solve for $Y(s)$.

$$Y(s) = \sum_{k=1}^{15} \frac{1}{s^2+1} e^{-(2k-1)\pi s}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

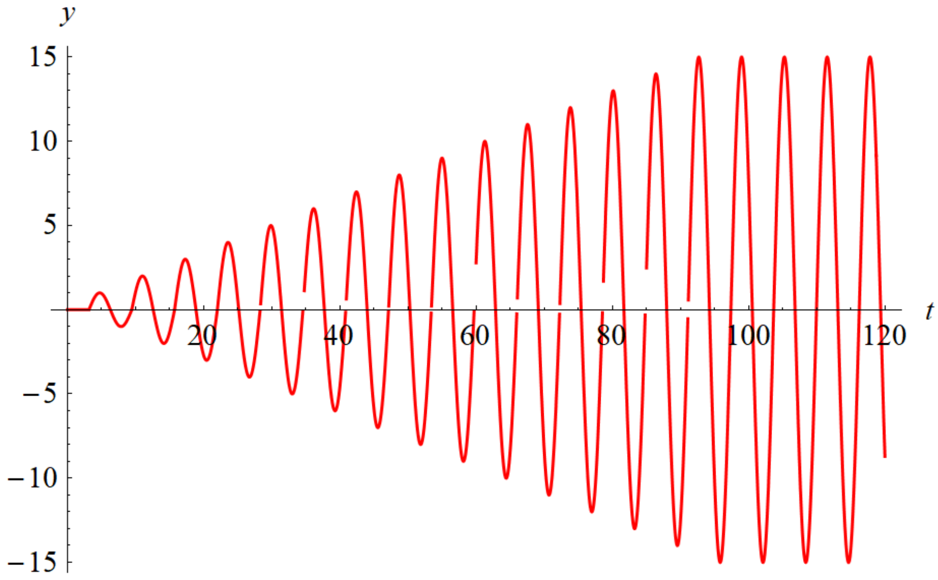
$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\sum_{k=1}^{15} \frac{1}{s^2+1} e^{-(2k-1)\pi s}\right\} \\ &= \sum_{k=1}^{15} \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} e^{-(2k-1)\pi s}\right\} \\ &= \sum_{k=1}^{15} \sin[t-(2k-1)\pi] H[t-(2k-1)\pi] \\ &= \sum_{k=1}^{15} \sin[t-(2k-1)\pi] u_{(2k-1)\pi}(t) \end{aligned}$$

Below is a plot of $y(t)$ versus t up until $t = 20$.



The delta functions strike upward with unit magnitude at the odd multiples of π, which is illustrated by the kinks in the graph.

Below is a plot of $y(t)$ versus t up until $t = 120$.



After $t = 29\pi \approx 91$, the delta function impulses stop, and the solution oscillates with constant amplitude from then on.