Problem 23

The position of a certain lightly damped oscillator satisfies the initial value problem

$$y'' + 0.1y' + y = \sum_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi), \qquad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 18.

(a) Try to predict the nature of the solution without solving the problem.

(b) Test your prediction by finding the solution and drawing its graph.

(c) Determine what happens after the sequence of impulses ends.

Solution

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + 0.1y' + y\} = \mathcal{L}\left\{\sum_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi)\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 0.1\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \sum_{k=1}^{20} (-1)^{k+1}\mathcal{L}\{\delta(t - k\pi)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 0.1[sY(s) - y(0)] + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} \int_0^\infty e^{-st}\delta(t - k\pi)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 0.1[sY(s)] + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-s(k\pi)}$$

$$\left(s^2 + \frac{1}{10}s + 1\right) Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-k\pi s}$$

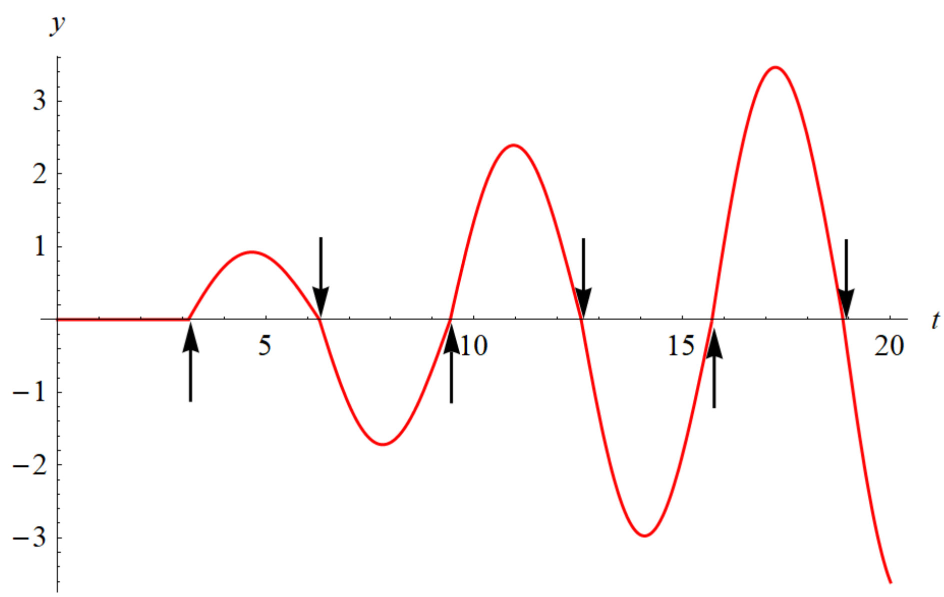
Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned} Y(s) &= \sum_{k=1}^{20}(-1)^{k+1}\frac{1}{s^2+\frac{1}{10}s+1}e^{-k\pi s} \\ &= \sum_{k=1}^{20}(-1)^{k+1}\frac{1}{s^2+\frac{1}{10}s+\frac{1}{400}+1-\frac{1}{400}}e^{-k\pi s} \\ &= \sum_{k=1}^{20}(-1)^{k+1}\frac{1}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}e^{-k\pi s} \\ &= \frac{20}{\sqrt{399}}\sum_{k=1}^{20}(-1)^{k+1}\frac{\frac{\sqrt{399}}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}e^{-k\pi s} \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

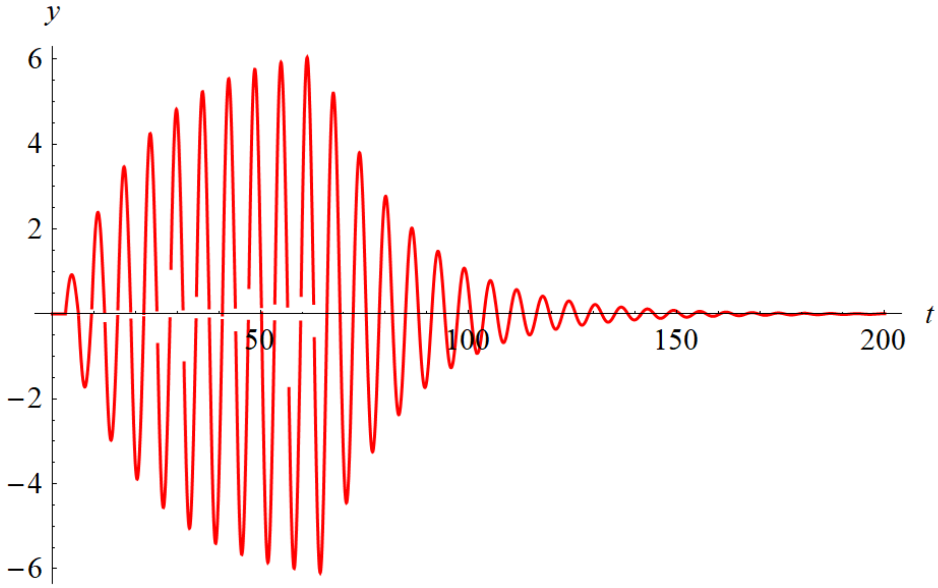
$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{20}{\sqrt{399}}\sum_{k=1}^{20}(-1)^{k+1}\frac{\frac{\sqrt{399}}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}e^{-k\pi s}\right\} \\ &= \frac{20}{\sqrt{399}}\sum_{k=1}^{20}(-1)^{k+1}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{399}}{20}}{\left(s+\frac{1}{20}\right)^2+\frac{399}{400}}e^{-k\pi s}\right\} \\ &= \frac{20}{\sqrt{399}}\sum_{k=1}^{20}(-1)^{k+1}e^{-(t-k\pi)/20}\sin\left[\frac{\sqrt{399}}{20}(t-k\pi)\right]H(t-k\pi) \\ &= \frac{20}{\sqrt{399}}\sum_{k=1}^{20}(-1)^{k+1}e^{-(t-k\pi)/20}\sin\left[\frac{\sqrt{399}}{20}(t-k\pi)\right]u_{k\pi}(t) \end{aligned}$$

Below is a plot of $y(t)$ versus t up until $t = 20$.



The arrows indicate the time, magnitude, and direction that the delta functions strike. Because of $(-1)^{k+1}$ in the inhomogeneous term, the direction of the delta function impulses alternates.

Below is a plot of $y(t)$ versus t up until $t = 200$.



After the delta function impulses stop, the amplitude of oscillation eventually decays to zero because of the y' term in the ODE.