Problem 8

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$y'' + 4y = 2\delta(t - \pi/4); \qquad y(0) = 0, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2 Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{2\delta(t - \pi/4)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 2\mathcal{L}\{\delta(t - \pi/4)\}$$

$$[s^2 Y(s) - sy(0) - y'(0)] + 4[Y(s)] = 2\int_0^\infty e^{-st}[\delta(t - \pi/4)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2 Y(s)] + 4[Y(s)] = 2e^{-s(\pi/4)}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 4)Y(s) = 2e^{-\pi s/4}$$

Solve for $Y(s)$.

$$Y(s) = \frac{2}{s^2 + 4} e^{-\pi s/4}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4} e^{-\pi s/4}\right\} \\ &= [\sin 2(t - \pi/4)]H(t - \pi/4) \\ &= \sin(2t - \pi/2)H(t - \pi/4) \\ &= (-\cos 2t)H(t - \pi/4) \\ &= -u_{\pi/4}(t)\cos 2t \end{aligned}$$

Below is a plot of $y(t)$ versus t.