Problem 11

In each of Problems 8 through 11, find the inverse Laplace transform of the given function by using the convolution theorem.

$$F(s) = \frac{G(s)}{s^2 + 1}$$

Solution

Recognize that $F(s)$ is a product of the two Laplace transforms,

$$G(s) = \mathcal{L}\{g(t)\} \quad \text{and} \quad \frac{1}{s^2+1} = \mathcal{L}\{\sin t\}.$$

According to the convolution theorem, the inverse Laplace transform of a product is a convolution integral.

$$\mathcal{L}^{-1}\{G(s)H(s)\} = \int_0^t g(t-\tau)h(\tau)\,d\tau$$

g and h are the inverse Laplace transforms of G and H, respectively. Therefore,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+1}\right\} \\ &= \int_0^t g(t-\tau)\sin\tau\,d\tau \\ &= \int_0^t \sin(t-\xi)g(\xi)\,d\xi. \end{aligned}$$