Problem 12

(a) If $f(t) = t^m$ and $g(t) = t^n$, where m and n are positive integers, show that

$$f * g = t^{m+n+1} \int_0^1 u^m (1-u)^n \, du.$$

(b) Use the convolution theorem to show that

$$\int_0^1 u^m (1-u)^n \, du = \frac{m!n!}{(m+n+1)!}.$$

(c) Extend the result of part (b) to the case where m and n are positive numbers but not necessarily integers.

Solution

Part (a)

Evaluate the convolution of f and g.

$$\begin{aligned} f * g &= \int_0^t f(t-\tau) g(\tau) \, d\tau \\ &= \int_0^t (t-\tau)^m (\tau)^n \, d\tau \\ &= \int_0^t \left[t\left(1-\frac{\tau}{t}\right)\right]^m \left[t\left(\frac{\tau}{t}\right)\right]^n \, d\tau \\ &= \int_0^t t^m \left(1-\frac{\tau}{t}\right)^m t^n \left(\frac{\tau}{t}\right)^n \, d\tau \\ &= t^{m+n} \int_0^t \left(1-\frac{\tau}{t}\right)^m \left(\frac{\tau}{t}\right)^n \, d\tau \end{aligned} \tag{1}$$

Make the substitution $u = \tau/t$. Then $du = d\tau/t$.

$$f * g = t^{m+n} \int_0^1 (1-u)^m u^n (t \, du)$$

Therefore,

$$f * g = t^{m+n+1} \int_0^1 u^m (1-u)^n \, du.$$

Part (b)

According to the convolution theorem, the inverse Laplace transform of a product $F(s)G(s)$ is a convolution integral.

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \mathcal{L}^{-1}\{F(s)G(s)\}$$

Use this result in equation (1).

$$\begin{aligned}
f*g &= \int_0^t (t-\tau)^m(\tau)^n\,d\tau \\
&= \mathcal{L}^{-1}\left\{\mathcal{L}\{t^m\}\mathcal{L}\{t^n\}\right\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{m!}{s^{m+1}}\right)\left(\frac{n!}{s^{n+1}}\right)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{m!n!}{s^{(m+n+1)+1}}\right\} \\
&= m!n!\mathcal{L}^{-1}\left\{\frac{1}{s^{(m+n+1)+1}}\right\} \\
&= \frac{m!n!}{(m+n+1)!}\mathcal{L}^{-1}\left\{\frac{(m+n+1)!}{s^{(m+n+1)+1}}\right\} \\
&= \frac{m!n!}{(m+n+1)!}t^{m+n+1}
\end{aligned}$$

Therefore, for positive integers,

$$t^{m+n+1}\int_0^1 u^m(1-u)^n\,du = \frac{m!n!}{(m+n+1)!}t^{m+n+1}$$

$$\int_0^1 u^m(1-u)^n\,du = \frac{m!n!}{(m+n+1)!}.$$

Part (c)

It was shown in part (c) of Problem 30 in Section 6.1 that $\Gamma(n+1) = n!$. The gamma function is an extension of the factorial function in the case that n is positive but not an integer. Use this result here.

$$\begin{aligned}
\int_0^1 u^m(1-u)^n\,du &= \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma[(m+n+1)+1]} \\
&= \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}
\end{aligned}$$