Problem 17

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

$$y'' + 4y' + 4y = g(t); \qquad y(0) = 2, \quad y'(0) = -3$$

Solution

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{g(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 4[Y(s)] = G(s)$$

Plug in the initial conditions, $y(0) = 2$ and $y'(0) = -3$.

$$[s^2Y(s) - 2s + 3] + 4[sY(s) - 2] + 4[Y(s)] = G(s)$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 4s + 4)Y(s) - 2s - 5 = G(s)$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{2s+5}{s^2+4s+4} + \frac{1}{s^2+4s+4}G(s) \\ &= \frac{2s+5}{(s+2)^2} + \frac{1}{(s+2)^2}G(s) \end{aligned}$$

Use partial fraction decomposition.

$$\frac{2s+5}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

Multiply both sides by $(s+2)^2$.

$$2s + 5 = A(s+2) + B$$

Plug in two random values of s to get a system of two equations for A and B.

$$\begin{aligned} s = 0: &\quad 5 = 2A + B \\ s = 1: &\quad 7 = 3A + B \end{aligned}$$

Solving this system yields $A = 2$ and $B = 1$.

$$Y(s) = \left[\frac{2}{s+2} + \frac{1}{(s+2)^2}\right] + \frac{1}{(s+2)^2}G(s)$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\left[\frac{2}{s+2} + \frac{1}{(s+2)^2}\right] + \frac{1}{(s+2)^2}G(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{s+2} + \frac{1}{(s+2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}G(s)\right\} \\ &= 2e^{-2t} + te^{-2t} + \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}G(s)\right\} \end{aligned}$$

Since we're taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Therefore,

$$y(t) = 2e^{-2t} + te^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)}g(\tau)\,d\tau.$$