Problem 21

Consider the equation

$$\phi(t) + \int_0^t k(t-\xi)\phi(\xi)\,d\xi = f(t),$$

in which f and k are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an **integral equation**; in particular, it belongs to a class of integral equations known as Volterra integral equations. Take the Laplace transform of the given integral equation and obtain an expression for $\mathcal{L}\{\phi(t)\}$ in terms of the transforms $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{k(t)\}$ of the given functions f and k. The inverse transform of $\mathcal{L}\{\phi(t)\}$ is the solution of the original integral equation.

Solution

The Laplace transform of a function $y(t)$ is defined as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt.$$

Consequently, the convolution theorem for it is

$$\mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)\,d\tau\right\} = F(s)G(s).$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\left\{\phi(t) + \int_0^t k(t-\xi)\phi(\xi)\,d\xi\right\} = \mathcal{L}\{f(t)\}$$

Use the fact that the transform is linear.

$$\mathcal{L}\{\phi(t)\} + \mathcal{L}\left\{\int_0^t k(t-\xi)\phi(\xi)\,d\xi\right\} = \mathcal{L}\{f(t)\}$$

Use the convolution theorem.

$$\mathcal{L}\{\phi(t)\} + \mathcal{L}\{k(t)\}\mathcal{L}\{\phi(t)\} = \mathcal{L}\{f(t)\}$$

Solve for $\mathcal{L}\{\phi(t)\}$.

$$\mathcal{L}\{\phi(t)\}\left(1 + \mathcal{L}\{k(t)\}\right) = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{\phi(t)\} = \frac{\mathcal{L}\{f(t)\}}{1 + \mathcal{L}\{k(t)\}}$$

Therefore,

$$\Phi(s) = \frac{F(s)}{1 + K(s)}.$$