Problem 22

Consider the Volterra integral equation (see Problem 21)

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = \sin 2t. \tag{i}$$

(a) Solve the integral equation (i) by using the Laplace transform.

(b) By differentiating Eq. (i) twice, show that $\phi(t)$ satisfies the differential equation

$$\phi''(t) + \phi(t) = -4\sin 2t.$$

Show also that the initial conditions are

$$\phi(0) = 0, \qquad \phi'(0) = 2.$$

(c) Solve the initial value problem in part (b), and verify that the solution is the same as the one in part (a).

Solution

Part (a)

The Laplace transform of a function $y(t)$ is defined as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt.$$

Consequently, the convolution theorem for it is

$$\mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)\,d\tau\right\} = F(s)G(s).$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\left\{\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi\right\} = \mathcal{L}\{\sin 2t\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{\phi(t)\} + \mathcal{L}\left\{\int_0^t (t-\xi)\phi(\xi)\,d\xi\right\} = \mathcal{L}\{\sin 2t\}$$

Apply the convolution theorem.

$$\mathcal{L}\{\phi(t)\} + \mathcal{L}\{t\}\mathcal{L}\{\phi(t)\} = \mathcal{L}\{\sin 2t\}$$

Evaluate the Laplace transforms.

$$\Phi(s) + \left(\frac{1}{s^2}\right)\Phi(s) = \frac{2}{s^2+4}$$

Solve for $\Phi(s)$.

$$\Phi(s)\left(1+\frac{1}{s^2}\right)=\frac{2}{s^2+4}$$

$$\Phi(s)=\frac{2s^2}{(s^2+1)(s^2+4)}$$

Use partial fraction decomposition to write $\Phi(s)$ in terms of known transforms.

$$\frac{2s^2}{(s^2+1)(s^2+4)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}$$

Multiply both sides by $(s^2+1)(s^2+4)$.

$$2s^2=(As+B)(s^2+4)+(Cs+D)(s^2+1)$$

Plug in four random values for s to obtain a system of four equations for A, B, C, and D.

$$\begin{aligned}
s=0: &\quad 0=4B+D \\
s=1: &\quad 2=5A+5B+2C+2D \\
s=2: &\quad 8=16A+8B+10C+5D \\
s=3: &\quad 18=39A+13B+30C+10D
\end{aligned}$$

Solving this system yields $A=0$, $B=-2/3$, $C=0$, and $D=8/3$.

$$\begin{aligned}
\Phi(s)&=\frac{-\frac{2}{3}}{s^2+1}+\frac{\frac{8}{3}}{s^2+4} \\
&=-\frac{2}{3}\frac{1}{s^2+1}+\frac{4}{3}\frac{2}{s^2+4}
\end{aligned}$$

Now take the inverse Laplace transform of $\Phi(s)$ to get $\phi(t)$.

$$\begin{aligned}
\phi(t)&=\mathcal{L}^{-1}\{\Phi(s)\} \\
&=\mathcal{L}^{-1}\left\{-\frac{2}{3}\frac{1}{s^2+1}+\frac{4}{3}\frac{2}{s^2+4}\right\} \\
&=-\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}+\frac{4}{3}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\
&=-\frac{2}{3}\sin t+\frac{4}{3}\sin 2t \\
&=\frac{2}{3}(-\sin t+2\sin 2t)
\end{aligned}$$

Part (b)

$$\phi(t)+\int_0^t (t-\xi)\phi(\xi)\,d\xi=\sin 2t \tag{i}$$

Plugging in $t=0$, we obtain the first initial condition for $\phi(t)$.

$$\phi(0)+\int_0^0 (-\xi)\phi(\xi)\,d\xi=0 \quad\rightarrow\quad \phi(0)=0$$

Differentiate both sides of Eq. (i) with respect to t.

$$\phi'(t) + \frac{d}{dt}\int_0^t (t-\xi)\phi(\xi)\,d\xi = 2\cos 2t$$

Apply the Leibnitz rule,

$$\frac{d}{dt}\int_{g(t)}^{h(t)} f(t,s)\,ds = \int_{g(t)}^{h(t)} \frac{\partial}{\partial t} f(t,s)\,ds - \frac{dg}{dt} f[t,g(t)] + \frac{dh}{dt} f[t,h(t)],$$

here to differentiate the integral.

$$\phi'(t) + \int_0^t \frac{\partial}{\partial t}(t-\xi)\phi(\xi)\,d\xi - 0\cdot(t)\phi(0) + 1\cdot(0)\phi(t) = 2\cos 2t$$

$$\phi'(t) + \int_0^t \phi(\xi)\,d\xi = 2\cos 2t \tag{ii}$$

Differentiate both sides with respect to t once more to obtain the ODE for $\phi(t)$.

$$\phi''(t) + \phi(t) = -4\sin 2t$$

Plug in $t = 0$ to Eq. (ii) to obtain the second initial condition for $\phi(t)$.

$$\phi'(0) + \int_0^0 \phi(\xi)\,d\xi = 2 \quad \rightarrow \quad \phi'(0) = 2$$

Part (c)

The initial value problem to solve here is

$$\phi''(t) + \phi(t) = -4\sin 2t, \qquad \phi(0) = 0, \quad \phi'(0) = 2.$$

This is a linear ODE, so its general solution can be expressed as a sum of the complementary solution and the particular solution.

$$\phi(t) = \phi_c(t) + \phi_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$\phi_c''(t) + \phi_c(t) = 0 \tag{1}$$

Since it has constant coefficients, the solution for it has the form $\phi_c = e^{rt}$.

$$\phi_c = e^{rt} \quad \rightarrow \quad \phi_c' = re^{rt} \quad \rightarrow \quad \phi_c'' = r^2 e^{rt}$$

Substitute these expressions into equation (1).

$$r^2 e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to equation (1) are then $\phi_c = e^{-it}$ and $\phi_c = e^{it}$. According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$\begin{aligned}\phi_c(t) &= C_1 e^{-it} + C_2 e^{it} \\ &= C_1(\cos t - i\sin t) + C_2(\cos t + i\sin t) \\ &= C_1\cos t - iC_1\sin t + C_2\cos t + iC_2\sin t \\ &= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t \\ &= C_3\cos t + C_4\sin t\end{aligned}$$

On the other hand, the particular solution satisfies

$$\phi_p''(t) + \phi_p(t) = -4\sin 2t. \tag{2}$$

Since there are only even derivatives on the left side and the right side consists of a sine term, the particular solution is of the form $\phi_p(t) = E\sin 2t$. Substitute this into equation (2) to determine E.

$$(E\sin 2t)'' + (E\sin 2t) = -4\sin 2t$$

Evaluate the derivatives.

$$-4E\sin 2t + E\sin 2t = -4\sin 2t$$

Divide both sides by $\sin 2t$.

$$\begin{aligned}-4E + E &= -4 \\ E &= \frac{4}{3}\end{aligned}$$

As a result, the particular solution is $\phi_p(t) = \frac{4}{3}\sin 2t$, and the general solution is

$$\begin{aligned}\phi(t) &= \phi_c(t) + \phi_p(t) \\ &= C_3\cos t + C_4\sin t + \frac{4}{3}\sin 2t.\end{aligned}$$

Take a derivative of it with respect to t.

$$\phi'(t) = -C_3\sin t + C_4\cos t + \frac{8}{3}\cos 2t$$

Now apply the initial conditions to determine C_3 and C_4.

$$\begin{aligned}\phi(0) &= C_3 = 0 \\ \phi'(0) &= C_4 + \frac{8}{3} = 2\end{aligned}$$

Solving this system yields $C_3 = 0$ and $C_4 = -2/3$. Therefore,

$$\begin{aligned}\phi(t) &= -\frac{2}{3}\sin t + \frac{4}{3}\sin 2t \\ &= \frac{2}{3}(-\sin t + 2\sin 2t).\end{aligned}$$