# Problem 26

There are also equations, known as **integro-differential equations**, in which both derivatives and integrals of the unknown function appear. In each of Problems 26 through 28:

(a) Solve the given integro-differential equation by using the Laplace transform.

(b) By differentiating the integro-differential equation a sufficient number of times, convert it into an initial value problem.

(c) Solve the initial value problem in part (b), and verify that the solution is the same as the one in part (a).

$$\phi'(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = t, \qquad \phi(0) = 0$$

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**Solution**

**Part (a)**

The Laplace transform of a function $y(t)$ is defined as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt.$$

Consequently, the first derivative transforms as

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

and the convolution theorem is

$$\mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)\,d\tau\right\} = F(s)G(s).$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\left\{\phi'(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi\right\} = \mathcal{L}\{t\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{\phi'(t)\} + \mathcal{L}\left\{\int_0^t (t-\xi)\phi(\xi)\,d\xi\right\} = \mathcal{L}\{t\}$$

Apply the convolution theorem.

$$\mathcal{L}\{\phi'(t)\} + \mathcal{L}\{t\}\mathcal{L}\{\phi(t)\} = \mathcal{L}\{t\}$$

Evaluate the Laplace transforms.

$$[s\Phi(s) - \phi(0)] + \left(\frac{1}{s^2}\right)\Phi(s) = \frac{1}{s^2}$$

Solve for $\Phi(s)$ and write it in terms of known transforms by using partial fraction decomposition.

$$\left(s+\frac{1}{s^2}\right)\Phi(s)=\frac{1}{s^2}$$

$$\frac{s^3+1}{s^2}\Phi(s)=\frac{1}{s^2}$$

$$\begin{aligned}
\Phi(s) &= \frac{1}{s^3+1} \\
&= \frac{1}{(s+1)(s^2-s+1)} \\
&= \frac{\frac{1}{3}}{s+1}+\frac{-\frac{1}{3}s+\frac{2}{3}}{s^2-s+1} \\
&= \frac{\frac{1}{3}}{s+1}+\frac{-\frac{1}{3}s+\frac{2}{3}}{s^2-s+\frac{1}{4}+1-\frac{1}{4}} \\
&= \frac{\frac{1}{3}}{s+1}+\frac{-\frac{1}{3}s+\frac{2}{3}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} \\
&= \frac{\frac{1}{3}}{s+1}+\frac{-\frac{1}{3}\left(s-\frac{1}{2}\right)-\frac{1}{6}+\frac{2}{3}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} \\
&= \frac{\frac{1}{3}}{s+1}+\frac{-\frac{1}{3}\left(s-\frac{1}{2}\right)+\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} \\
&= \frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} \\
&= \frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}
\end{aligned}$$

Now take the inverse Laplace transform of $\Phi(s)$ to get $\phi(t)$.

$$\begin{aligned}
\phi(t) &= \mathcal{L}^{-1}\{\Phi(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\} \\
&= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}+\frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\} \\
&= \frac{1}{3}e^{-t}-\frac{1}{3}e^{t/2}\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}e^{t/2}\sin\frac{\sqrt{3}}{2}t
\end{aligned}$$

## Part (b)

$$\phi'(t) + \int_0^t (t - \xi)\phi(\xi)\, d\xi = t, \qquad \phi(0) = 0$$

Differentiate both sides of the integral equation with respect to $t$.

$$\phi''(t) + \frac{d}{dt}\int_0^t (t - \xi)\phi(\xi)\, d\xi = 1$$

Apply the Leibnitz rule,

$$\frac{d}{dt}\int_{g(t)}^{h(t)} f(t,s)\, ds = \int_{g(t)}^{h(t)} \frac{\partial}{\partial t} f(t,s)\, ds - \frac{dg}{dt} f[t, g(t)] + \frac{dh}{dt} f[t, h(t)],$$

here to differentiate the integral.

$$\phi''(t) + \int_0^t \frac{\partial}{\partial t}(t - \xi)\phi(\xi)\, d\xi - 0 \cdot (t)\phi(0) + 1 \cdot (0)\phi(t) = 1$$

$$\phi''(t) + \int_0^t \phi(\xi)\, d\xi = 1 \tag{1}$$

Differentiate both sides with respect to $t$ once more.

$$\phi'''(t) + \phi(t) = 0 \tag{2}$$

Plug in $t = 0$ to the original integral equation and equation (1) to obtain the second and third initial conditions for $\phi(t)$.

$$\phi'(0) + \int_0^0 (-\xi)\phi(\xi)\, d\xi = 0 \qquad \to \qquad \phi'(0) = 0$$

$$\phi''(0) + \int_0^0 \phi(\xi)\, d\xi = 1 \qquad \to \qquad \phi''(0) = 1$$

## Part (c)

Since equation (2) is a linear homogeneous ODE with constant coefficients, the solution is of the form $\phi = e^{rt}$.

$$\phi = e^{rt} \quad \to \quad \phi' = re^{rt} \quad \to \quad \phi'' = r^2 e^{rt} \quad \to \quad \phi''' = r^3 e^{rt}$$

Substitute these expressions into equation (2).

$$r^3 e^{rt} + e^{rt} = 0$$

Divide both sides by $e^{rt}$.

$$r^3 + 1 = 0$$

$$\begin{aligned} r &= \sqrt[3]{-1} \\ &= \sqrt[3]{e^{i\pi + 2in\pi}}, \quad n = 0, \pm 1, \pm 2, \ldots \\ &= e^{i\pi/3 + 2in\pi/3} \end{aligned}$$

The three distinct roots are obtained by setting $n = 0$, $n = 1$, and $n = 2$. Other values of $n$ result in redundant roots.

$$\begin{aligned}
n = 0: &\quad r_1 = e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\
n = 1: &\quad r_2 = e^{i\pi} = \cos\pi + i\sin\pi = -1 \\
n = 2: &\quad r_3 = e^{5i\pi/3} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}
\end{aligned}$$

Three solutions to equation (2) are then $\phi = e^{(1/2+i\sqrt{3}/2)t}$, $\phi = e^{-t}$, and $\phi = e^{(1/2-i\sqrt{3}/2)t}$. According to the principle of superposition, the general solution is a linear combination of these three.

$$\begin{aligned}
\phi(t) &= C_1 e^{(1/2+i\sqrt{3}/2)t} + C_2 e^{-t} + C_3 e^{(1/2-i\sqrt{3}/2)t} \\
&= C_1 e^{t/2+i\sqrt{3}t/2} + C_2 e^{-t} + C_3 e^{t/2-i\sqrt{3}t/2} \\
&= C_1 e^{t/2} e^{i\sqrt{3}t/2} + C_2 e^{-t} + C_3 e^{t/2} e^{-i\sqrt{3}t/2} \\
&= C_1 e^{t/2}\left(\cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t\right) + C_2 e^{-t} + C_3 e^{t/2}\left(\cos\frac{\sqrt{3}}{2}t - i\sin\frac{\sqrt{3}}{2}t\right) \\
&= e^{t/2}\left[(C_1 + C_3)\cos\frac{\sqrt{3}}{2}t + (iC_1 - iC_3)\sin\frac{\sqrt{3}}{2}t\right] + C_2 e^{-t} \\
&= e^{t/2}\left(C_4\cos\frac{\sqrt{3}}{2}t + C_5\sin\frac{\sqrt{3}}{2}t\right) + C_2 e^{-t}
\end{aligned}$$

Differentiate it with respect to $t$ twice.

$$\begin{aligned}
\phi'(t) &= \frac{1}{2}e^{t/2}\left(C_4\cos\frac{\sqrt{3}}{2}t + C_5\sin\frac{\sqrt{3}}{2}t\right) + e^{t/2}\left(-\frac{\sqrt{3}}{2}C_4\sin\frac{\sqrt{3}}{2}t + C_5\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t\right) - C_2 e^{-t} \\
\phi''(t) &= \frac{1}{4}e^{t/2}\left(C_4\cos\frac{\sqrt{3}}{2}t + C_5\sin\frac{\sqrt{3}}{2}t\right) + \frac{1}{2}e^{t/2}\left(-C_4\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t + C_5\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t\right) \\
&\quad + \frac{1}{2}e^{t/2}\left(-\frac{\sqrt{3}}{2}C_4\sin\frac{\sqrt{3}}{2}t + C_5\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t\right) + e^{t/2}\left(-\frac{3}{4}C_4\cos\frac{\sqrt{3}}{2}t - C_5\frac{3}{4}\sin\frac{\sqrt{3}}{2}t\right) + C_2 e^{-t}
\end{aligned}$$

Now apply the initial conditions to determine $C_2$, $C_4$, and $C_5$.

$$\begin{aligned}
\phi(0) &= C_4 + C_2 = 0 \\
\phi'(0) &= \frac{1}{2}C_4 + C_5\frac{\sqrt{3}}{2} - C_2 = 0 \\
\phi''(0) &= \frac{1}{4}C_4 + C_5\frac{\sqrt{3}}{4} + C_5\frac{\sqrt{3}}{4} - \frac{3}{4}C_4 + C_2 = 1
\end{aligned}$$

Solving this system yields $C_2 = 1/3$, $C_4 = -1/3$, and $C_5 = 1/\sqrt{3}$. Therefore,

$$\begin{aligned}
\phi(t) &= e^{t/2}\left(-\frac{1}{3}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right) + \frac{1}{3}e^{-t} \\
&= \frac{1}{3}e^{-t} - \frac{1}{3}e^{t/2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}e^{t/2}\sin\frac{\sqrt{3}}{2}t.
\end{aligned}$$