Problem 10

In each of Problems 8 through 11, find the inverse Laplace transform of the given function by using the convolution theorem.

$$F(s) = \frac{1}{(s+1)^2(s^2+4)}$$

Solution

Recognize that $F(s)$ is a product of the two Laplace transforms,

$$\frac{1}{(s+1)^2} = \mathcal{L}\{te^{-t}\} \quad \text{and} \quad \frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+4} = \mathcal{L}\left\{\frac{1}{2}\sin 2t\right\}.$$

According to the convolution theorem, the inverse Laplace transform of a product is a convolution integral.

$$\mathcal{L}^{-1}\left\{G(s)H(s)\right\} = \int_0^t g(t-\tau)h(\tau)\,d\tau$$

g and h are the inverse Laplace transforms of G and H, respectively. Therefore,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\} \\ &= \int_0^t (t-\tau)e^{-(t-\tau)}\frac{1}{2}\sin 2\tau\,d\tau. \end{aligned}$$