# Problem 15

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

$$4y'' + 4y' + 17y = g(t); \qquad y(0) = 0, \quad y'(0) = 0$$

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**Solution**

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{4y'' + 4y' + 17y\} = \mathcal{L}\{g(t)\}$$

Use the fact that the transform is a linear operator.

$$4\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 17\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$4[s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 17[Y(s)] = G(s)$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$4[s^2Y(s)] + 4[sY(s)] + 17[Y(s)] = G(s)$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for $Y$, the transformed solution.

$$(4s^2 + 4s + 17)Y(s) = G(s)$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned}
Y(s) &= \frac{1}{4s^2 + 4s + 17}G(s) \\
&= \frac{1}{4}\frac{1}{s^2 + s + \frac{17}{4}}G(s) \\
&= \frac{1}{4}\frac{1}{s^2 + s + \frac{1}{4} + \frac{17}{4} - \frac{1}{4}}G(s) \\
&= \frac{1}{4}\frac{1}{\left(s + \frac{1}{2}\right)^2 + 4}G(s) \\
&= \frac{1}{8}\frac{2}{\left(s + \frac{1}{2}\right)^2 + 4}G(s)
\end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{8}\frac{2}{\left(s+\frac{1}{2}\right)^2+4}G(s)\right\} \\ &= \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{2}{\left(s+\frac{1}{2}\right)^2+4}G(s)\right\} \end{aligned}$$

Since we're taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Therefore,

$$y(t) = \frac{1}{8}\int_0^t e^{-(t-\tau)/2}\sin[2(t-\tau)]g(\tau)\,d\tau.$$