Problem 16

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

$$y'' + y' + \frac{5}{4}y = 1 - u_\pi(t); \qquad y(0) = 1, \quad y'(0) = -1$$

Solution

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\left\{y'' + y' + \frac{5}{4}y\right\} = \mathcal{L}\{1 - u_\pi(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \frac{5}{4}\mathcal{L}\{y\} = \mathcal{L}\{1 - u_\pi(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + \frac{5}{4}[Y(s)] = \mathcal{L}\{1 - u_\pi(t)\}$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = -1$.

$$[s^2Y(s) - s + 1] + [sY(s) - 1] + \frac{5}{4}[Y(s)] = \mathcal{L}\{1 - u_\pi(t)\}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$\left(s^2 + s + \frac{5}{4}\right)Y(s) - s = \mathcal{L}\{1 - u_\pi(t)\}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned}
Y(s) &= \frac{s}{s^2 + s + \frac{5}{4}} + \frac{1}{s^2 + s + \frac{5}{4}}\mathcal{L}\{1 - u_\pi(t)\} \\
&= \frac{s}{s^2 + s + \frac{1}{4} + \frac{5}{4} - \frac{1}{4}} + \frac{1}{s^2 + s + \frac{1}{4} + \frac{5}{4} - \frac{1}{4}}\mathcal{L}\{1 - u_\pi(t)\} \\
&= \frac{s}{\left(s + \frac{1}{2}\right)^2 + 1} + \frac{1}{\left(s + \frac{1}{2}\right)^2 + 1}\mathcal{L}\{1 - u_\pi(t)\}
\end{aligned}$$

Make it so that $s+\frac{1}{2}$ appears in the numerator of the first term.

$$\begin{aligned} Y(s) &= \frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} + \frac{1}{\left(s+\frac{1}{2}\right)^2+1}\mathcal{L}\{1-u_\pi(t)\} \\ &= \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} - \frac{1}{2}\frac{1}{\left(s+\frac{1}{2}\right)^2+1} + \frac{1}{\left(s+\frac{1}{2}\right)^2+1}\mathcal{L}\{1-u_\pi(t)\} \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} - \frac{1}{2}\frac{1}{\left(s+\frac{1}{2}\right)^2+1} + \frac{1}{\left(s+\frac{1}{2}\right)^2+1}\mathcal{L}\{1-u_\pi(t)\}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\mathcal{L}\{1-u_\pi(t)\}\right\} \\ &= e^{-t/2}\cos t - \frac{1}{2}e^{-t/2}\sin t + \mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\mathcal{L}\{1-u_\pi(t)\}\right\} \end{aligned}$$

Since we're taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Therefore,

$$y(t) = e^{-t/2}\cos t - \frac{1}{2}e^{-t/2}\sin t + \int_0^t e^{-(t-\tau)/2}\sin(t-\tau)[1-u_\pi(\tau)]\,d\tau.$$