Problem 19

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

$$y^{(4)} - y = g(t); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$$

Solution

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the derivatives transform as follows.

$$\begin{aligned}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)
\end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y^{(4)} - y\} = \mathcal{L}\{g(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] - [Y(s)] = G(s)$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$ and $y''(0) = 0$ and $y'''(0) = 0$.

$$[s^4Y(s)] - [Y(s)] = G(s)$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^4 - 1)Y(s) = G(s)$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned}
Y(s) &= \frac{1}{s^4 - 1} G(s) \\
&= \frac{1}{(s-1)(s+1)(s^2+1)} G(s) \\
&= \left(\frac{\frac{1}{4}}{s-1} - \frac{\frac{1}{4}}{s+1} - \frac{\frac{1}{2}}{s^2+1}\right) G(s)
\end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{4}}{s-1} - \frac{\frac{1}{4}}{s+1} - \frac{\frac{1}{2}}{s^2+1}\right)G(s)\right\} \end{aligned}$$

Since we're taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Therefore,

$$\begin{aligned} y(t) &= \int_0^t \left[\frac{1}{4}e^{(t-\tau)} - \frac{1}{4}e^{-(t-\tau)} - \frac{1}{2}\sin(t-\tau)\right] g(\tau)\,d\tau \\ &= \int_0^t \left[\frac{1}{2}\frac{e^{(t-\tau)} - e^{-(t-\tau)}}{2} - \frac{1}{2}\sin(t-\tau)\right] g(\tau)\,d\tau \\ &= \int_0^t \left[\frac{1}{2}\sinh(t-\tau) - \frac{1}{2}\sin(t-\tau)\right] g(\tau)\,d\tau \\ &= \frac{1}{2}\int_0^t [\sinh(t-\tau) - \sin(t-\tau)]g(\tau)\,d\tau. \end{aligned}$$