Problem 20

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

$$y^{(4)} + 5y'' + 4y = g(t); \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$$

Solution

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the derivatives transform as follows.

$$\begin{aligned}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)
\end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y^{(4)} + 5y'' + 4y\} = \mathcal{L}\{g(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y^{(4)}\} + 5\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] + 5[s^2Y(s) - sy(0) - y'(0)] + 4[Y(s)] = G(s)$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = 0$ and $y''(0) = 0$ and $y'''(0) = 0$.

$$[s^4Y(s) - s^3] + 5[s^2Y(s) - s] + 4[Y(s)] = G(s)$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^4 + 5s^2 + 4)Y(s) - s^3 - 5s = G(s)$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned}
Y(s) &= \frac{s^3 + 5s}{s^4 + 5s^2 + 4} + \frac{1}{s^4 + 5s^2 + 4}G(s) \\
&= \frac{s^3 + 5s}{(s^2 + 4)(s^2 + 1)} + \frac{1}{(s^2 + 4)(s^2 + 1)}G(s)
\end{aligned}$$

Use partial fraction decomposition.

$$\begin{aligned} Y(s) &= \frac{\frac{4}{3}s}{s^2+1} + \frac{-\frac{1}{3}s}{s^2+4} + \left(\frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4}\right)G(s) \\ &= \frac{\frac{4}{3}s}{s^2+1} + \frac{-\frac{1}{3}s}{s^2+4} + \left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right)G(s) \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{\frac{4}{3}s}{s^2+1} + \frac{-\frac{1}{3}s}{s^2+4} + \left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right)G(s)\right\} \\ &= \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right)G(s)\right\} \\ &= \frac{4}{3}\cos t - \frac{1}{3}\cos 2t + \mathcal{L}^{-1}\left\{\left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right)G(s)\right\} \end{aligned}$$

Since we're taking the inverse Laplace transform of a product of two transforms, we can use the convolution theorem. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{4}{3}\cos t - \frac{1}{3}\cos 2t + \int_0^t \left\{\frac{1}{3}\sin(t-\tau) - \frac{1}{6}\sin[2(t-\tau)]\right\} g(\tau)\,d\tau \\ &= \frac{1}{3}(4\cos t - \cos 2t) + \frac{1}{6}\int_0^t \{2\sin(t-\tau) - \sin[2(t-\tau)]\}\, g(\tau)\,d\tau. \end{aligned}$$