Problem 23

In each of Problems 23 through 25:

(a) Solve the given Volterra integral equation by using the Laplace transform.

(b) Convert the integral equation into an initial value problem, as in Problem 22(b).

(c) Solve the initial value problem in part (b), and verify that the solution is the same as the one in part (a).

$$\phi(t) + \int_0^t (t - \xi)\phi(\xi)\, d\xi = 1$$

Solution

Part (a)

The Laplace transform of a function $y(t)$ is defined as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the convolution theorem for it is

$$\mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)\, d\tau\right\} = F(s)G(s).$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\left\{\phi(t) + \int_0^t (t-\xi)\phi(\xi)\, d\xi\right\} = \mathcal{L}\{1\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{\phi(t)\} + \mathcal{L}\left\{\int_0^t (t-\xi)\phi(\xi)\, d\xi\right\} = \mathcal{L}\{1\}$$

Apply the convolution theorem.

$$\mathcal{L}\{\phi(t)\} + \mathcal{L}\{t\}\mathcal{L}\{\phi(t)\} = \mathcal{L}\{1\}$$

Evaluate the Laplace transforms.

$$\Phi(s) + \left(\frac{1}{s^2}\right)\Phi(s) = \frac{1}{s}$$

Solve for $\Phi(s)$.

$$\Phi(s)\left(1 + \frac{1}{s^2}\right) = \frac{1}{s}$$

$$\Phi(s) = \frac{s}{s^2+1}$$

Now take the inverse Laplace transform of $\Phi(s)$ to get $\phi(t)$.

$$\begin{aligned}\phi(t) &= \mathcal{L}^{-1}\{\Phi(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \\ &= \cos t\end{aligned}$$

Part (b)

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = 1$$

Plug in $t = 0$ to the integral equation to obtain the first initial condition for $\phi(t)$.

$$\phi(0) + \int_0^0 (-\xi)\phi(\xi)\,d\xi = 1 \quad \rightarrow \quad \phi(0) = 1$$

Differentiate both sides of the integral equation with respect to t.

$$\phi'(t) + \frac{d}{dt}\int_0^t (t-\xi)\phi(\xi)\,d\xi = 0$$

Apply the Leibnitz rule,

$$\frac{d}{dt}\int_{g(t)}^{h(t)} f(t,s)\,ds = \int_{g(t)}^{h(t)} \frac{\partial}{\partial t} f(t,s)\,ds - \frac{dg}{dt} f[t, g(t)] + \frac{dh}{dt} f[t, h(t)],$$

here to differentiate the integral.

$$\phi'(t) + \int_0^t \frac{\partial}{\partial t}(t-\xi)\phi(\xi)\,d\xi - 0\cdot(t)\phi(0) + 1\cdot(0)\phi(t) = 0$$

$$\phi'(t) + \int_0^t \phi(\xi)\,d\xi = 0$$

Plug in $t = 0$ to get the second initial condition for $\phi(t)$.

$$\phi'(0) + \int_0^0 \phi(\xi)\,d\xi = 0 \quad \rightarrow \quad \phi'(0) = 0$$

Differentiate both sides of the previous equation with respect to t once more to obtain the ODE for $\phi(t)$.

$$\phi''(t) + \phi(t) = 0$$

Part (c)

Since it has constant coefficients, the solution for it has the form $\phi = e^{rt}$.

$$\phi = e^{rt} \quad \rightarrow \quad \phi' = re^{rt} \quad \rightarrow \quad \phi'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to equation (1) are then $\phi = e^{-it}$ and $\phi = e^{it}$. According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$\begin{aligned}
\phi(t) &= C_1 e^{-it} + C_2 e^{it} \\
&= C_1(\cos t - i \sin t) + C_2(\cos t + i \sin t) \\
&= C_1 \cos t - iC_1 \sin t + C_2 \cos t + iC_2 \sin t \\
&= (C_1 + C_2) \cos t + (-iC_1 + iC_2) \sin t \\
&= C_3 \cos t + C_4 \sin t
\end{aligned}$$

Differentiate it with respect to t.

$$\phi'(t) = -C_3 \sin t + C_4 \cos t$$

Now apply the initial conditions to determine C_3 and C_4.

$$\begin{aligned}
\phi(0) &= C_3 = 1 \\
\phi'(0) &= C_4 = 0
\end{aligned}$$

Therefore,

$$\phi(t) = \cos t.$$