Problem 25

In each of Problems 23 through 25:

(a) Solve the given Volterra integral equation by using the Laplace transform.

(b) Convert the integral equation into an initial value problem, as in Problem 22(b).

(c) Solve the initial value problem in part (b), and verify that the solution is the same as the one in part (a).

$$\phi(t) + 2\int_0^t \cos(t-\xi)\phi(\xi)\,d\xi = e^{-t}$$

Solution

Part (a)

The Laplace transform of a function $y(t)$ is defined as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt.$$

Consequently, the convolution theorem for it is

$$\mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)\,d\tau\right\} = F(s)G(s).$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\left\{\phi(t) + 2\int_0^t \cos(t-\xi)\phi(\xi)\,d\xi\right\} = \mathcal{L}\{e^{-t}\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{\phi(t)\} + 2\mathcal{L}\left\{\int_0^t \cos(t-\xi)\phi(\xi)\,d\xi\right\} = \mathcal{L}\{e^{-t}\}$$

Apply the convolution theorem.

$$\mathcal{L}\{\phi(t)\} + 2\mathcal{L}\{\cos t\}\mathcal{L}\{\phi(t)\} = \mathcal{L}\{e^{-t}\}$$

Evaluate the Laplace transforms.

$$\Phi(s) + 2\left(\frac{s}{s^2+1}\right)\Phi(s) = \frac{1}{s+1}$$

Solve for $\Phi(s)$.

$$\Phi(s)\left(1+\frac{2s}{s^2+1}\right) = \frac{1}{s+1}$$

$$\Phi(s)\frac{s^2+2s+1}{s^2+1} = \frac{1}{s+1}$$

$$\Phi(s) = \frac{s^2+1}{(s+1)^3}$$

Write it in terms of known transforms by using partial fraction decomposition.

$$\frac{s^2+1}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

Multiply both sides by $(s+1)^3$.

$$s^2+1 = A(s+1)^2 + B(s+1) + C$$

Plug in three random values for s to obtain a system of equations for A, B, and C.

$$\begin{aligned} s=0: &\quad 1 = A+B+C \\ s=1: &\quad 2 = 4A+2B+C \\ s=2: &\quad 5 = 9A+3B+C \end{aligned}$$

Solving this system yields $A=1$, $B=-2$, and $C=2$.

$$\Phi(s) = \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}$$

Now take the inverse Laplace transform of $\Phi(s)$ to get $\phi(t)$.

$$\begin{aligned} \phi(t) &= \mathcal{L}^{-1}\{\Phi(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^3}\right\} \\ &= e^{-t} - 2te^{-t} + t^2e^{-t} \end{aligned}$$

Part (b)

$$\phi(t) + 2\int_0^t \cos(t-\xi)\phi(\xi)\,d\xi = e^{-t}$$

Plug in $t=0$ to the integral equation to obtain the first initial condition for $\phi(t)$.

$$\phi(0) + 2\int_0^0 \cos(-\xi)\phi(\xi)\,d\xi = e^{-0} \quad \rightarrow \quad \phi(0) = 1$$

Differentiate both sides of the integral equation with respect to t.

$$\phi'(t) + 2\frac{d}{dt}\int_0^t \cos(t-\xi)\phi(\xi)\,d\xi = -e^{-t}$$

Apply the Leibnitz rule,

$$\frac{d}{dt}\int_{g(t)}^{h(t)} f(t,s)\,ds = \int_{g(t)}^{h(t)} \frac{\partial}{\partial t} f(t,s)\,ds - \frac{dg}{dt} f[t,g(t)] + \frac{dh}{dt} f[t,h(t)],$$

here to differentiate the integral.

$$\phi'(t) + 2\left[\int_0^t \frac{\partial}{\partial t}\cos(t-\xi)\phi(\xi)\,d\xi - 0\cdot\cos(t)\phi(0) + 1\cdot\cos(0)\phi(t)\right] = -e^{-t}$$

Simplify the left side.

$$\phi'(t) + 2\int_0^t [-\sin(t-\xi)]\phi(\xi)\,d\xi + 2\phi(t) = -e^{-t}$$

Plug in $t = 0$ to get the second initial condition for $\phi(t)$.

$$\phi'(0) + 2\int_0^0 [-\sin(-\xi)]\phi(\xi)\,d\xi + 2\phi(0) = -e^{-0} \quad \rightarrow \quad \phi'(0) + 2\phi(0) = -1 \quad \rightarrow \quad \phi'(0) = -3$$

Differentiate both sides of the previous equation with respect to t once more.

$$\phi''(t) + 2\frac{d}{dt}\int_0^t [-\sin(t-\xi)]\phi(\xi)\,d\xi + 2\phi'(t) = e^{-t}$$

$$\phi''(t) + 2\left\{\int_0^t \frac{\partial}{\partial t}[-\sin(t-\xi)]\phi(\xi)\,d\xi - 0\cdot(-\sin t)\phi(0) + 1\cdot(-\sin 0)\phi(t)\right\} + 2\phi'(t) = e^{-t}$$

$$\phi''(t) + 2\left\{\int_0^t [-\cos(t-\xi)]\phi(\xi)\,d\xi\right\} + 2\phi'(t) = e^{-t}$$

$$\phi''(t) - 2\int_0^t \cos(t-\xi)\phi(\xi)\,d\xi + 2\phi'(t) = e^{-t}$$

Solve the original integral equation for this integral

$$\phi(t) + 2\int_0^t \cos(t-\xi)\phi(\xi)\,d\xi = e^{-t} \quad \rightarrow \quad \int_0^t \cos(t-\xi)\phi(\xi)\,d\xi = \frac{1}{2}[e^{-t} - \phi(t)]$$

and then substitute it into the equation.

$$\phi''(t) - 2\frac{1}{2}[e^{-t} - \phi(t)] + 2\phi'(t) = e^{-t}$$

$$\phi''(t) + 2\phi'(t) + \phi(t) = 2e^{-t}$$

Part (c)

Since this ODE is linear, its general solution can be expressed as a sum of the complementary solution and the particular solution.

$$\phi(t) = \phi_c(t) + \phi_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$\phi_c''(t) + 2\phi_c'(t) + \phi_c(t) = 0 \tag{1}$$

It has constant coefficients, so the solution for it has the form $\phi_c = e^{rt}$.

$$\phi_c = e^{rt} \quad \rightarrow \quad \phi_c' = re^{rt} \quad \rightarrow \quad \phi_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + 2(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r = \{-1\}$$

One solution to equation (1) is then $\phi_c = e^{-t}$. The multiplicity of the $r = -1$ root is 2, so a second solution can be obtained from the first by including a factor of t: $\phi_c = te^{-t}$. According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$\phi_c(t) = C_1 e^{-t} + C_2 t e^{-t}$$

On the other hand, the particular solution satisfies

$$\phi_p''(t) + 2\phi_p'(t) + \phi_p(t) = 2e^{-t}. \tag{2}$$

It would have the form $\phi_p = De^{-t}$, but since e^{-t} satisfies equation (1), it will be te^{-t}. Actually, because te^{-t} satisfies equation (1) as well, it's going to be $\phi_p = Dt^2e^{-t}$. Substitute it into equation (2) to determine D.

$$(Dt^2e^{-t})'' + 2(Dt^2e^{-t})' + (Dt^2e^{-t}) = 2e^{-t}$$

Simplify the left side.

$$2De^{-t} = 2e^{-t}$$

$$D = 1$$

So then $\phi_p(t) = t^2e^{-t}$, and the general solution for $\phi(t)$ is

$$\begin{aligned}\phi(t) &= \phi_c(t) + \phi_p(t)\\ &= C_1e^{-t} + C_2te^{-t} + t^2e^{-t}.\end{aligned}$$

Differentiate it with respect to t.

$$\phi'(t) = -C_1e^{-t} + C_2e^{-t} - C_2te^{-t} + 2te^{-t} - t^2e^{-t}$$

Now apply the initial conditions to determine C_1 and C_2.

$$\begin{aligned}\phi(0) &= C_1 = 1\\ \phi'(0) &= -C_1 + C_2 = -3\end{aligned}$$

Solving this system yields $C_1 = 1$ and $C_2 = -2$. Therefore,

$$\phi(t) = e^{-t} - 2te^{-t} + t^2e^{-t}.$$