Problem 29

The Tautochrone. A problem of interest in the history of mathematics is that of finding the tautochrone[6]—the curve down which a particle will slide freely under gravity alone, reaching the bottom in the same time regardless of its starting point on the curve. This problem arose in the construction of a clock pendulum whose period is independent of the amplitude of its motion. The tautochrone was found by Christian Huygens (1629–1695) in 1673 by geometrical methods, and later by Leibniz and Jakob Bernoulli using analytical arguments. Bernoulli's solution (in 1690) was one of the first occasions on which a differential equation was explicitly solved. The geometric configuration is shown in Figure 6.6.2. The starting point $P(a,b)$ is joined to the terminal point $(0,0)$ by the arc C. Arc length s is measured from the origin, and $f(y)$ denotes the rate of change of s with respect to y:

$$f(y) = \frac{ds}{dy} = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2}. \tag{i}$$

Then it follows from the principle of conservation of energy that the time $T(b)$ required for a particle to slide from P to the origin is

$$T(b) = \frac{1}{\sqrt{2g}}\int_0^b \frac{f(y)}{\sqrt{b-y}}\,dy. \tag{ii}$$

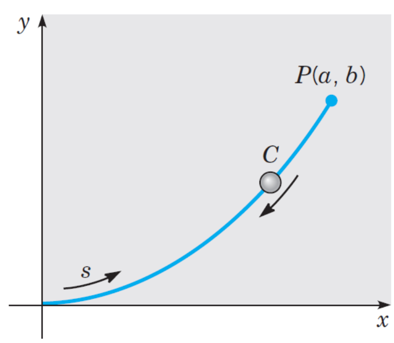

FIGURE 6.6.2 The tautochrone.

(a) Assume that $T(b) = T_0$, a constant, for each b. By taking the Laplace transform of Eq. (ii) in this case, and using the convolution theorem, show that

$$F(s) = \sqrt{\frac{2g}{\pi}}\frac{T_0}{\sqrt{s}}; \tag{iii}$$

then show that

$$f(y) = \frac{\sqrt{2g}}{\pi}\frac{T_0}{\sqrt{y}}. \tag{iv}$$

Hint: See Problem 31 of Section 6.1.

[6] The word "tautochrone" comes from the Greek words *tauto*, which means "same," and *chronos*, which means "time."

(b) Combining Eqs. (i) and (iv), show that

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}, \tag{v}$$

where $\alpha = gT_0^2/\pi^2$.

(c) Use the substitution $y = 2\alpha \sin^2(\theta/2)$ to solve Eq. (v), and show that

$$x = \alpha(\theta + \sin\theta), \qquad y = \alpha(1 - \cos\theta). \tag{vi}$$

Equations (vi) can be identified as parametric equations of a cycloid. Thus the tautochrone is an arc of a cycloid.

Solution

Here we define the Laplace transform of a function $f(b)$ to be

$$F(s) = \mathcal{L}\{f(b)\} = \int_0^\infty e^{-sb} f(b)\, db.$$

Consequently, the convolution theorem is

$$\mathcal{L}\left\{\int_0^b f(b-y)g(y)\, dy\right\} = F(s)G(s).$$

Part (a)

Take the Laplace transform of both sides of Eq. (ii).

$$\mathcal{L}\{T(b)\} = \mathcal{L}\left\{\frac{1}{\sqrt{2g}}\int_0^b \frac{f(y)}{\sqrt{b-y}}\, dy\right\}$$

Substitute $T(b) = T_0$ and bring the constants out.

$$T_0\mathcal{L}\{1\} = \frac{1}{\sqrt{2g}}\mathcal{L}\left\{\int_0^b \frac{f(y)}{\sqrt{b-y}}\, dy\right\}$$

Apply the convolution theorem.

$$T_0\mathcal{L}\{1\} = \frac{1}{\sqrt{2g}}\mathcal{L}\left\{\frac{1}{\sqrt{b}}\right\}\mathcal{L}\{f(b)\}$$

Evaluate the Laplace transforms and then solve for $F(s)$.

$$T_0\frac{1}{s} = \frac{1}{\sqrt{2g}}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}\right]F(s)$$

$$\frac{T_0}{s} = \frac{1}{\sqrt{2g}}\left(\frac{\sqrt{\pi}}{\sqrt{s}}\right)F(s)$$

$$F(s) = \sqrt{\frac{2g}{\pi}}\frac{T_0}{\sqrt{s}}$$

Now take the inverse Laplace transform of $F(s)$ to get $f(b)$.

$$\begin{aligned}
f(b) &= \mathcal{L}^{-1}\{F(s)\} \\
&= \mathcal{L}^{-1}\left\{\sqrt{\frac{2g}{\pi}}\frac{T_0}{\sqrt{s}}\right\} \\
&= \mathcal{L}^{-1}\left\{T_0\frac{\sqrt{2g}}{\pi}\frac{\sqrt{\pi}}{\sqrt{s}}\right\} \\
&= T_0\frac{\sqrt{2g}}{\pi}\mathcal{L}^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{s}}\right\} \\
&= T_0\frac{\sqrt{2g}}{\pi}\mathcal{L}^{-1}\left\{\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}\right\} \\
&= T_0\frac{\sqrt{2g}}{\pi}b^{-1/2} \\
&= \frac{\sqrt{2g}}{\pi}\frac{T_0}{\sqrt{b}}
\end{aligned}$$

Therefore, changing the variable to y,

$$f(y) = \frac{\sqrt{2g}}{\pi}\frac{T_0}{\sqrt{y}}.$$

Part (b)

Substitute this result for $f(y)$ into Eq. (i).

$$f(y) = \frac{ds}{dy} = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2} = \frac{\sqrt{2g}}{\pi}\frac{T_0}{\sqrt{y}}$$

Square both sides.

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{2g}{\pi^2}\frac{T_0^2}{y}$$

Let $\alpha = gT_0^2/\pi^2$.

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{2\alpha}{y}$$

$$\begin{aligned}
\left(\frac{dx}{dy}\right)^2 &= \frac{2\alpha}{y} - 1 \\
&= \frac{2\alpha - y}{y}
\end{aligned}$$

Take the square root of both sides.

$$\frac{dx}{dy} = \pm\sqrt{\frac{2\alpha - y}{y}}$$

We choose the positive sign because the curve lies in the first quadrant of the xy-plane.

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}$$

Part (c)

Make the substitution $y = 2\alpha \sin^2(\theta/2)$.

$$\begin{aligned} \frac{dx}{dy} &= \sqrt{\frac{2\alpha - 2\alpha \sin^2 \frac{\theta}{2}}{2\alpha \sin^2 \frac{\theta}{2}}} \\ &= \sqrt{\frac{1 - \sin^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}}} \\ &= \sqrt{\frac{\cos^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}}} \\ &= \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \end{aligned}$$

The positive root is taken here again because the curve lies in the first quadrant of the xy-plane. Use the chain rule find what dx/dy is in terms of this new variable.

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{dx}{dy}\frac{dy}{d\theta} \\ &= \frac{dx}{dy}\left(4\alpha \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cdot \frac{1}{2}\right) \\ &= \frac{dx}{dy}\left(2\alpha \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right) \end{aligned}$$

Divide both sides by $2\alpha \sin \frac{\theta}{2} \cos \frac{\theta}{2}$.

$$\frac{dx}{dy} = \frac{1}{2\alpha \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \frac{dx}{d\theta}$$

Substitute this result for dx/dy in the ODE.

$$\frac{1}{2\alpha \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \frac{dx}{d\theta} = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}$$

Solve for $dx/d\theta$.

$$\begin{aligned} \frac{dx}{d\theta} &= 2\alpha \cos^2 \frac{\theta}{2} \\ &= 2\alpha \frac{1}{2}(1 + \cos\theta) \\ &= \alpha(1 + \cos\theta) \end{aligned}$$

Integrate both sides with respect to θ, setting the integration constant to zero because the curve starts at the origin.

$$x(\theta) = \alpha(\theta + \sin\theta)$$

Finally, use the trigonometric identity $2\sin^2 \frac{\theta}{2} = 1 - \cos\theta$ in the initial substitution to obtain $y(\theta)$.

$$\begin{aligned} y(\theta) &= 2\alpha \sin^2 \frac{\theta}{2} \\ &= \alpha(1 - \cos\theta) \end{aligned}$$