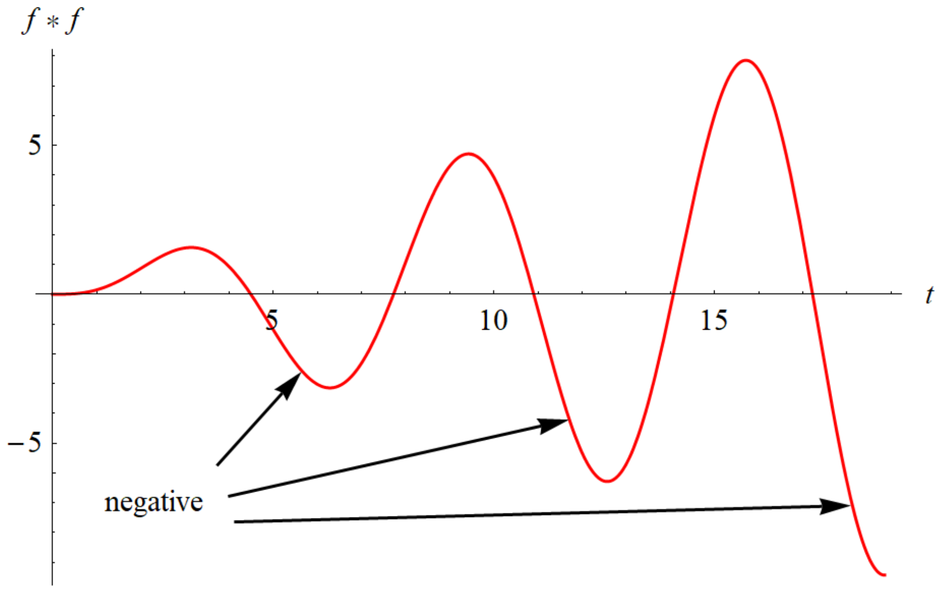
Problem 3

Show, by means of the example $f(t) = \sin t$, that $f * f$ is not necessarily nonnegative.

Solution

Evaluate the convolution integral $f * f$.

$$
\begin{aligned}
f * f &= \int_0^t f(t-\tau)f(\tau)\,d\tau \\
&= \int_0^t \sin(t-\tau)\sin(\tau)\,d\tau \\
&= \int_0^t (\sin t\cos\tau - \cos t\sin\tau)\sin\tau\,d\tau \\
&= \int_0^t (\sin t\sin\tau\cos\tau - \cos t\sin^2\tau)\,d\tau \\
&= \sin t\int_0^t \sin\tau\cos\tau\,d\tau - \cos t\int_0^t \sin^2\tau\,d\tau \\
&= \sin t\int_0^t \frac{1}{2}\sin 2\tau\,d\tau - \cos t\int_0^t \frac{1}{2}(1-\cos 2\tau)\,d\tau \\
&= \sin t\left(-\frac{1}{4}\cos 2\tau\right)\bigg|_0^t - \frac{1}{2}\cos t\left(\tau - \frac{1}{2}\sin 2\tau\right)\bigg|_0^t \\
&= \sin t\left(-\frac{1}{4}\cos 2t + \frac{1}{4}\right) - \frac{1}{2}\cos t\left(t - \frac{1}{2}\sin 2t\right) \\
&= -\frac{1}{4}\sin t(\cos 2t - 1) - \frac{1}{2}t\cos t + \frac{1}{4}\cos t\sin 2t \\
&= -\frac{1}{4}\sin t(-2\sin^2 t) - \frac{1}{2}t\cos t + \frac{1}{2}\cos^2 t\sin t \\
&= \frac{1}{2}\sin^3 t - \frac{1}{2}t\cos t + \frac{1}{2}(1-\sin^2 t)\sin t \\
&= -\frac{1}{2}t\cos t + \frac{1}{2}\sin t \\
&= \frac{1}{2}(\sin t - t\cos t)
\end{aligned}
$$



As the graph of $(\sin t - t\cos t)/2$ versus t shows, $f * f$ is not necessarily nonnegative.