Exercise 2

In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:

$$(a)\ (-16)^{1/4}; \qquad (b)\ (-8-8\sqrt{3}i)^{1/4}.$$

Ans. $(a)\ \pm\sqrt{2}(1+i),\ \pm\sqrt{2}(1-i);\quad (b)\ \pm(\sqrt{3}-i),\ \pm(1+\sqrt{3}i).$

Solution

For a nonzero complex number $z = re^{i(\Theta+2\pi k)}$, its fourth roots are

$$z^{1/4} = \left[re^{i(\Theta+2\pi k)}\right]^{1/4} = r^{1/4}\exp\left(i\frac{\Theta+2\pi k}{4}\right), \quad k = 0, 1, 2, 3.$$

Part (a)

The magnitude of -16 is 16, and the principal argument is π.

$$(-16)^{1/4} = 16^{1/4}\exp\left(i\frac{\pi+2\pi k}{4}\right), \quad k = 0, 1, 2, 3$$

The first, or principal, root ($k = 0$) is

$$(-16)^{1/4} = 16^{1/4}e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1+i),$$

the second root ($k = 1$) is

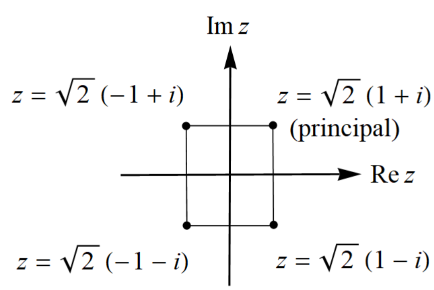
$$(-16)^{1/4} = 16^{1/4}e^{i3\pi/4} = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(-1+i),$$

the third root ($k = 2$) is

$$(-16)^{1/4} = 16^{1/4}e^{i5\pi/4} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(-1-i),$$

and the fourth root ($k = 3$) is

$$(-16)^{1/4} = 16^{1/4}e^{i7\pi/4} = 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1-i).$$



Part (b)

The magnitude and principal argument of $-8-8\sqrt{3}i$ are respectively

$$r=\sqrt{(-8)^2+(-8\sqrt{3})^2}=16 \quad \text{and} \quad \Theta=\tan^{-1}\frac{-8\sqrt{3}}{-8}-\pi=-\frac{2\pi}{3},$$

so

$$(-8-8\sqrt{3}i)^{1/4}=16^{1/4}\exp\left(i\frac{-\frac{2\pi}{3}+2\pi k}{4}\right), \quad k=0,1,2,3.$$

The first, or principal, root ($k=0$) is

$$(-8-8\sqrt{3}i)^{1/4}=16^{1/4}e^{-i\pi/6}=2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)=2\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=\sqrt{3}-i,$$

the second root ($k=1$) is

$$(-8-8\sqrt{3}i)^{1/4}=16^{1/4}e^{i\pi/3}=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=1+\sqrt{3}i,$$

the third root ($k=2$) is

$$(-8-8\sqrt{3}i)^{1/4}=16^{1/4}e^{i5\pi/6}=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=2\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=-\sqrt{3}+i,$$

and the fourth root ($k=3$) is

$$(-8-8\sqrt{3}i)^{1/4}=16^{1/4}e^{i4\pi/3}=2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)=2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=-1-\sqrt{3}i.$$