Exercise 5

(a) Let a denote any fixed real number and show that the two square roots of $a+i$ are

$$\pm\sqrt{A}\exp\left(i\frac{\alpha}{2}\right)$$

where $A=\sqrt{a^2+1}$ and $\alpha=\operatorname{Arg}(a+i)$.

(b) With the aid of the trigonometric identities (4) in Example 3 of Sec. 10, show that the square roots obtained in part (a) can be written

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a}+i\sqrt{A-a}\right).$$

(Note that this becomes the final result in Example 3, Sec. 10, when $a=\sqrt{3}$.)

Solution

Part (a)

For a nonzero complex number $z=re^{i(\Theta+2\pi k)}$, its square roots are

$$z^{1/2}=\left[re^{i(\Theta+2\pi k)}\right]^{1/2}=r^{1/2}\exp\left(i\frac{\Theta+2\pi k}{2}\right),\quad k=0,1.$$

The magnitude and principal argument of $a+i$ are respectively

$$r=\sqrt{a^2+1^2}=\sqrt{a^2+1}\quad\text{and}\quad\Theta=\operatorname{Arg}(a+i),$$

so

$$(a+i)^{1/2}=\left(\sqrt{a^2+1}\right)^{1/2}\exp\left(i\frac{\operatorname{Arg}(a+i)+2\pi k}{2}\right)=\sqrt{A}\exp\left(i\frac{\alpha+2\pi k}{2}\right)$$
$$=\sqrt{A}\exp\left(i\frac{\alpha}{2}\right)e^{i\pi k},\quad k=0,1.$$

The first root ($k=0$) is

$$(a+i)^{1/2}=\sqrt{A}\exp\left(i\frac{\alpha}{2}\right),$$

and the second root ($k=1$) is

$$(a+i)^{1/2}=\sqrt{A}\exp\left(i\frac{\alpha}{2}\right)e^{i\pi}=\sqrt{A}\exp\left(i\frac{\alpha}{2}\right)(\cos\pi+i\sin\pi)=\sqrt{A}\exp\left(i\frac{\alpha}{2}\right)(-1+i0)$$
$$=-\sqrt{A}\exp\left(i\frac{\alpha}{2}\right).$$

Part (b)

The trigonometric identities (4) in Example 3 of Sec. 10 are

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}, \quad \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}. \tag{4}$$

Take the square roots of both sides of each equation.

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}, \quad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Since a is real, $\alpha = \operatorname{Arg}(a+i)$ is either in the first or second quadrant $(0 < \alpha < \pi)$. This means that $\alpha/2$ is in the first quadrant, so the positive signs are chosen.

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}, \quad \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$$

The square roots of $(a+i)^{1/2}$ become

$$\begin{aligned}
(a+i)^{1/2} &= \pm\sqrt{A}\exp\left(i\frac{\alpha}{2}\right) \\
&= \pm\sqrt{A}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right) \\
&= \pm\sqrt{A}\left(\sqrt{\frac{1+\cos\alpha}{2}} + i\sqrt{\frac{1-\cos\alpha}{2}}\right) \\
&= \pm\frac{1}{\sqrt{2}}\left(\sqrt{A + A\cos\alpha} + i\sqrt{A - A\cos\alpha}\right).
\end{aligned}$$

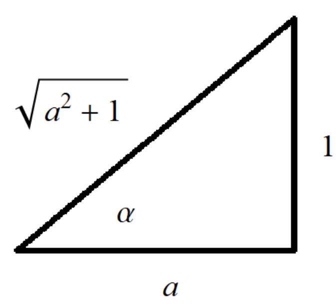
Suppose first that a is positive. Then

$$\alpha = \operatorname{Arg}(a+i) = \tan^{-1}\frac{1}{a}$$

and

$$\cos\alpha = \cos\tan^{-1}\frac{1}{a}.$$

Draw the implied right triangle to determine the cosine.



As a result,

$$\cos\alpha = \frac{a}{\sqrt{a^2+1}} = \frac{a}{A} \quad \rightarrow \quad A\cos\alpha = a$$

and

$$(a+i)^{1/2} = \pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a} + i\sqrt{A-a}\right).$$

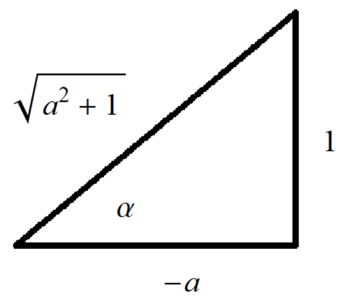
Suppose secondly that a is negative. Then

$$\alpha = \operatorname{Arg}(a+i) = \tan^{-1}\frac{1}{a} + \pi$$

and

$$\cos\alpha = \cos\left(\tan^{-1}\frac{1}{a} + \pi\right) = \cos\left(-\tan^{-1}\frac{1}{a} - \pi\right) = \cos\left(\tan^{-1}\frac{1}{-a} - \pi\right) = -\cos\tan^{-1}\frac{1}{-a}.$$

Draw the implied right triangle to determine the cosine.



As a result,

$$\cos\alpha = -\left(\frac{-a}{\sqrt{a^2+1}}\right) = \frac{a}{\sqrt{a^2+1}} = \frac{a}{A} \quad \rightarrow \quad A\cos\alpha = a$$

and

$$(a+i)^{1/2} = \pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a} + i\sqrt{A-a}\right).$$

This same result holds regardless of whether a is positive or negative.