

## Exercise 14

Using expressions (6), Sec. 5, for  $\operatorname{Re} z$  and  $\operatorname{Im} z$ , show that the hyperbola  $x^2 - y^2 = 1$  can be written

$$z^2 + \bar{z}^2 = 2.$$

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### Solution

$x$  and  $y$  are the real and imaginary parts of a complex number  $z = x + iy$ , respectively. If we have

$$x^2 - y^2 = 1,$$

then

$$\begin{aligned}(\operatorname{Re} z)^2 - (\operatorname{Im} z)^2 &= 1 \\ \left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 &= 1 \\ \frac{1}{4}(z + \bar{z})^2 - \frac{1}{4i^2}(z - \bar{z})^2 &= 1 \\ \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) + \frac{1}{4}(z^2 - 2z\bar{z} + \bar{z}^2) &= 1.\end{aligned}$$

Multiply both sides by 4.

$$\begin{aligned}(z^2 + 2z\bar{z} + \bar{z}^2) + (z^2 - 2z\bar{z} + \bar{z}^2) &= 4 \\ 2z^2 + 2\bar{z}^2 &= 4\end{aligned}$$

Therefore, the hyperbola  $x^2 - y^2 = 1$  can be written as

$$z^2 + \bar{z}^2 = 2.$$