Exercise 10

The integration formula

$$\int_0^\infty \frac{dx}{[(x^2-a)^2+1]^2} = \frac{\pi}{8\sqrt{2}A^3}[(2a^2+3)\sqrt{A+a}+a\sqrt{A-a}],$$

where a is any real number and $A=\sqrt{a^2+1}$, arises in the theory of case-hardening of steel by means of radio-frequency heating.* Follow the steps below to derive it.

(a) Point out why the four zeros of the polynomial

$$q(z) = (z^2-a)^2+1$$

are the square roots of the numbers $a \pm i$. Then, using the fact that the numbers

$$z_0 = \frac{1}{\sqrt{2}}(\sqrt{A+a}+i\sqrt{A-a})$$

and $-z_0$ are the square roots of $a+i$ (Exercise 5, Sec. 10), verify that $\pm\overline{z_0}$ are the square roots of $a-i$ and hence that z_0 and $-\overline{z_0}$ are the only zeros of $q(z)$ in the upper half plane $\operatorname{Im} z \geq 0$.

(b) Using the method derived in Exercise 7, Sec. 76, and keeping in mind that $z_0^2 = a+i$ for purposes of simplification, show that the point z_0 in part (a) is a pole of order 2 of the function $f(z) = 1/[q(z)]^2$ and that the residue B_1 at z_0 can be written

$$B_1 = -\frac{q''(z_0)}{[q'(z_0)]^3} = \frac{a-i(2a^2+3)}{16A^2z_0}.$$

After observing that $q'(-\bar{z}) = -\overline{q'(z)}$ and $q''(-\bar{z}) = \overline{q''(z)}$, use the same method to show that the point $-\overline{z_0}$ in part (a) is also a pole of order 2 of the function $f(z)$, with residue

$$B_2 = \overline{\left\{\frac{q''(z_0)}{[q'(z_0)]^3}\right\}} = -\overline{B_1}.$$

Then obtain the expression

$$B_1 + B_2 = \frac{1}{8A^2i}\operatorname{Im}\left[\frac{-a+i(2a^2+3)}{z_0}\right]$$

for the sum of the residues.

(c) Refer to part (a) and show that $|q(z)| \geq (R-|z_0|)^4$ if $|z| = R$, where $R > |z_0|$. Then, with the aid of the final result in part (b), complete the derivation of the integration formula.

*See pp. 359–364 of the book by Brown, Hoyler, and Bierwirth that is listed in Appendix 1.