Exercise 4

Use residues to derive the integration formulas in Exercises 1 through 6.

$$\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}.$$

Solution

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{x^2\,dx}{x^6+1} = \int_{-\infty}^\infty \frac{x^2\,dx}{2(x^6+1)}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{z^2}{2(z^6+1)},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$2(z^6+1) = 0$$
$$z^6+1 = 0$$

$$z = \sqrt[6]{1}\exp\left[i\left(\frac{\pi+2k\pi}{6}\right)\right], \quad k = 0, 1, \ldots, 5 \quad \rightarrow \quad \begin{cases} z_1 = e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{i}{2} \\ z_2 = e^{i3\pi/6} = i \\ z_3 = e^{i5\pi/6} = -\frac{\sqrt{3}}{2} + \frac{i}{2} \\ z_4 = e^{i7\pi/6} = -\frac{\sqrt{3}}{2} - \frac{i}{2} \\ z_5 = e^{i9\pi/6} = -i \\ z_6 = e^{i11\pi/6} = \frac{\sqrt{3}}{2} - \frac{i}{2} \end{cases}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = z_1$ and $z = z_2$ and $z = z_3$.

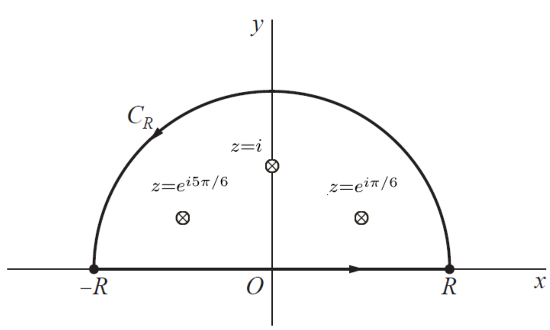

Figure 1: This is Fig. 93 with the singularities at $z = z_1$ and $z = z_2$ and $z = z_3$ marked.

According to Cauchy's residue theorem, the integral of $z^2/[2(z^6+1)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^2\,dz}{2(z^6+1)} = 2\pi i\left[\operatorname*{Res}_{z=z_1}\frac{z^2}{2(z^6+1)} + \operatorname*{Res}_{z=z_2}\frac{z^2}{2(z^6+1)} + \operatorname*{Res}_{z=z_3}\frac{z^2}{2(z^6+1)}\right]$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{z^2\,dz}{2(z^6+1)} + \int_{C_R} \frac{z^2\,dz}{2(z^6+1)} = 2\pi i\left[\operatorname*{Res}_{z=z_1}\frac{z^2}{2(z^6+1)} + \operatorname*{Res}_{z=z_2}\frac{z^2}{2(z^6+1)} + \operatorname*{Res}_{z=z_3}\frac{z^2}{2(z^6+1)}\right]$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, && r=-R \quad\to\quad r=R \\ C_R: &\quad z = Re^{i\theta}, && \theta=0 \quad\to\quad \theta=\pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{r^2\,dr}{2(r^6+1)} + \int_{C_R} \frac{z^2\,dz}{2(z^6+1)} = 2\pi i\left[\operatorname*{Res}_{z=z_1}\frac{z^2}{2(z^6+1)} + \operatorname*{Res}_{z=z_2}\frac{z^2}{2(z^6+1)} + \operatorname*{Res}_{z=z_3}\frac{z^2}{2(z^6+1)}\right].$$

Take the limit now as $R\to\infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{r^2\,dr}{2(r^6+1)} = 2\pi i\left[\operatorname*{Res}_{z=z_1}\frac{z^2}{2(z^6+1)} + \operatorname*{Res}_{z=z_2}\frac{z^2}{2(z^6+1)} + \operatorname*{Res}_{z=z_3}\frac{z^2}{2(z^6+1)}\right]$$

The denominator can be written as $2(z^6+1) = 2(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)$. From this we see that the multiplicities of the $z-z_1$ and $z-z_2$ and $z-z_3$ factors are all 1. The residues at $z=z_1$, $z=z_2$, and $z=z_3$ can then be calculated by

$$\begin{aligned} \operatorname*{Res}_{z=z_1}\frac{z^2}{2(z^6+1)} &= \phi_1(z_1) \\ \operatorname*{Res}_{z=z_2}\frac{z^2}{2(z^6+1)} &= \phi_2(z_2) \\ \operatorname*{Res}_{z=z_3}\frac{z^2}{2(z^6+1)} &= \phi_3(z_3), \end{aligned}$$

where $\phi_1(z)$, $\phi_2(z)$, and $\phi_3(z)$ are equal to $f(z)$ without the $z-z_1$, $z-z_2$, and $z-z_3$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{z^2}{2(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)} &\Rightarrow&\quad \phi_1(z_1) = -\frac{i}{12} \\ \phi_2(z) &= \frac{z^2}{2(z-z_1)(z-z_3)(z-z_4)(z-z_5)(z-z_6)} &\Rightarrow&\quad \phi_2(z_2) = \frac{i}{12} \\ \phi_3(z) &= \frac{z^2}{2(z-z_1)(z-z_2)(z-z_4)(z-z_5)(z-z_6)} &\Rightarrow&\quad \phi_3(z_3) = -\frac{i}{12} \end{aligned}$$

So then

$$\operatorname*{Res}_{z=z_1} \frac{z^2}{2(z^6+1)} = -\frac{i}{12}$$
$$\operatorname*{Res}_{z=z_2} \frac{z^2}{2(z^6+1)} = \frac{i}{12}$$
$$\operatorname*{Res}_{z=z_3} \frac{z^2}{2(z^6+1)} = -\frac{i}{12}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{r^2\,dr}{2(r^6+1)} &= 2\pi i \left(-\frac{i}{12} + \frac{i}{12} - \frac{i}{12}\right) \\ &= 2\pi i \left(-\frac{i}{12}\right) \\ &= \frac{\pi}{6}.\end{aligned}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty} \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{z^2\,dz}{2(z^6+1)} &= \int_0^\pi \frac{(Re^{i\theta})^2 Rie^{i\theta}\,d\theta}{2[(Re^{i\theta})^6+1]} \\ &= \int_0^\pi \frac{R^3 ie^{i3\theta}}{R^6e^{i6\theta}+1}\,\frac{d\theta}{2}\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{z^2\,dz}{2(z^6+1)}\right| &= \left|\int_0^\pi \frac{R^3 ie^{i3\theta}}{R^6e^{i6\theta}+1}\,\frac{d\theta}{2}\right| \\ &\le \int_0^\pi \left|\frac{R^3 ie^{i3\theta}}{R^6e^{i6\theta}+1}\right|\frac{d\theta}{2} \\ &= \int_0^\pi \frac{\left|R^3 ie^{i3\theta}\right|}{|R^6e^{i6\theta}+1|}\,\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R^3}{|R^6e^{i6\theta}+1|}\,\frac{d\theta}{2} \\ &\le \int_0^\pi \frac{R^3}{|R^6e^{i6\theta}|-|1|}\,\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R^3}{R^6-1}\,\frac{d\theta}{2} \\ &= \frac{\pi}{2}\frac{R^3}{R^6-1}\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{z^2\,dz}{2(z^6+1)}\right| &\le \lim_{R\to\infty} \frac{\pi}{2}\frac{R^3}{R^6-1} \\ &= \lim_{R\to\infty} \frac{\pi}{2R^3}\frac{1}{1-\frac{1}{R^6}}\end{aligned}$$

The limit on the right side is zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^2\,dz}{2(z^6+1)}\right| \le 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^2\,dz}{2(z^6+1)}\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{z^2\,dz}{2(z^6+1)}\,dz = 0.$$